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A NOTE ON HILBERT TRANSFORMS

By H. KOBER (*Birmingham*)

[Received 31 December 1941; in revised form 3 April 1942]

1. Let $1 < p < \infty$ and $f(t) \in L_p$ over $(-\infty, \infty)$, and let $\mathfrak{H}f$ be Hilbert's operator

$$\mathfrak{H}[f;x] = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(t)\,dt}{t-x} = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{f(x+t)-f(x-t)}{t}\,dt.$$

Then (A) *$\mathfrak{H}f$ exists for almost all x in $(-\infty, \infty)$.*

(B) $|\mathfrak{H}f|_p \leqslant C_p |f|_p$,† *where C_p depends on p only, and $C_2 = 1$.*

(C) *$\mathfrak{H}f = g$ implies $\mathfrak{H}g = -f$: that is to say, $\mathfrak{H}^2 f = -f$.*

(D) $$\int_{-\infty}^{\infty} f_1(x)\mathfrak{H}f_2\,dx = -\int_{-\infty}^{\infty} f_2(x)\mathfrak{H}f_1\,dx$$

$$(f_1 \in L_p,\ f_2 \in L_{p'};\ 1/p+1/p' = 1).$$

(E) $\mathfrak{H}[f(t-a); x] = \mathfrak{H}[f(t); x-a]$ $(a \text{ real})$.

The theory of Hilbert transforms has been developed by M. Riesz, G. H. Hardy, and E. C. Titchmarsh.‡ In the present paper I shall give a new method of proving (A)–(E), based upon the approximation to integrable functions by rational functions. This method exhibits clearly the properties (C) and (D) of $\mathfrak{H}f$ and gives some new results on the characteristic functions.

By Q we denote the set of rational functions which are regular on the real axis and vanish at infinity; by E or E^* respectively the set of rational functions $R(z)$ or $R^*(z)$ $(z = x+iy)$ which vanish at infinity and are regular for $y \geqslant 0$ or $y \leqslant 0$. Evidently a function $S(z)$ belongs to Q if and only if it is representable in the form

$$S(z) = R(z)+R^*(z), \qquad R(z) \in E, \qquad R^*(z) \in E^*. \tag{1.1}$$

My thanks are due to the referees for suggesting some alteration in the paper.

† There is equality when $p = 2$. By $|F(t)|_p$ or $|F|_p$ we denote

$$\left(\int_{-\infty}^{\infty} |F(t)|^p\,dt\right)^{1/p}.$$

‡ G. H. Hardy ($p = 2$), *Quart J. of Math.* (Oxford) 3 (1932), 102–12. E. C. Titchmarsh, *Math. Zeitschr.* 25 (1926), 321–47. M. Riesz, *Math. Zeitschr.* 27 (1928), 218–44. Cf. J. Cossar, *Proc. London Math. Soc.* (2) 45 (1939), 369–81.

2. Approximation by rational functions. I shall prove

THEOREM 1. *Let* (i) *a be any positive integer and*

$$r_{n,a}(z) = (i-z)^n(i+z)^{-n-a},$$

(ii) $G(t)(1+|t|)^{-1} \in L_1(-\infty, \infty)$,

(iii) $$\int\limits_{-\infty}^{\infty} G(t)r_{n,a}(t)\,dt = 0 \quad (n = 0, \pm 1, \pm 2,...). \tag{2.1}$$

Then $$G(t) \equiv 0 \textit{ in } (-\infty, \infty).$$

Proof. Taking $t = \tan\frac{1}{2}\theta$ $(-\pi < \theta < \pi)$, we put

$$G(t) = g(\theta)e^{-ia\theta/2}(\cos\tfrac{1}{2}\theta)^{2-a}.$$

Then (ii) implies that $g(\theta)$ is integrable over $(-\pi, \pi)$, and (2.1) is transformed into

$$\tfrac{1}{2}i^{-a}\int\limits_{-\pi}^{\pi} g(\theta)e^{in\theta}\,d\theta = 0 \quad (n = 0, \pm 1, \pm 2,...).$$

By the well-known completeness of the sequence $\{e^{in\theta}\}$ with respect to the space $L(-\pi, \pi)$, we have $g(\theta) \equiv 0$, which proves the theorem.

COROLLARY 1. *Replacing* (ii) *by the condition*

(ii′) $$G(t) \in L_p(-\infty, \infty) \quad (1 \leqslant p \leqslant \infty),$$

the theorem holds, if $a \geqslant 0$ *for* $p = 1$, $a \geqslant 2$ *for* $p = \infty$, *and* $a \geqslant 1$ *for* $1 < p < \infty$.

The case $1 < p < \infty$ is obvious. When $p = 1$ or $p = \infty$, the result is deduced by applying the theorem to $G(t)(i+t)$ or $G(t)(i+t)^{-1}$ respectively.

COROLLARY 2. *The sequence*

$$\{\pi^{-\frac{1}{2}}(i-t)^n(i+t)^{-n-1}\} \quad (n = 0, \pm 1, \pm 2,...)$$

is a complete orthogonal and normal system with respect to

$$L_p(-\infty, \infty) \quad (1 \leqslant p < \infty).$$

COROLLARY 3. *Let* $a \geqslant 2$ *for* $p = 1$ *and* $a \geqslant 1$ *for* $1 < p < \infty$. *Given* $F(t) \in L_p(-\infty, \infty)$ $(1 \leqslant p < \infty)$, *there are rational functions* $S_n(z)$, *represented in the form*

$$S_n(z) = \sum_{k=-n}^{n} a_{k,n}\frac{(i-z)^k}{(i+z)^{k+a}} \quad (n = 0, 1, 2,...)$$

and such that $|F(t)-S_n(t)|_p \to 0$ *as* $n \to \infty$.

This is deduced from the corollary 1 by a well-known theorem.†

It can be shown that the theorem, and therefore the corollaries 1 and 3, hold when we take $r_{n,a}(z) = (\alpha - z)^n(z-\beta)^{-n-a}$ where α and β are arbitrary numbers such that $\mathfrak{I}(\alpha) > 0, \mathfrak{I}(\beta) < 0$.

3. To deal with Hilbert's operator we start with

LEMMA 1. *Let $R(z) \in E$, $R^*(z) \in E^*$. Then*

$$\mathfrak{H}R = iR, \qquad \mathfrak{H}R^* = -iR^*.$$

Proof. Let Γ be a contour consisting of the two semicircles $\zeta = x+\epsilon e^{i\phi}$, $\zeta = Me^{i\phi}$ $(0 \leqslant \phi \leqslant \pi;\ 0 < \epsilon < M-|x|)$ and the two segments $-M \leqslant \zeta \leqslant x-\epsilon$, $x+\epsilon \leqslant \zeta \leqslant M$ of the real axis, and let R belong to E. Then

$$\left(\int\limits_{-M}^{x-\epsilon} + \int\limits_{x+\epsilon}^{M}\right)\frac{R(t)\,dt}{t-x} - i\int\limits_{0}^{\pi} R(x+\epsilon e^{i\phi})\,d\phi + i\int\limits_{0}^{\pi}\frac{Me^{i\phi}R(Me^{i\phi})\,d\phi}{Me^{i\phi}-x}$$

$$= \int\limits_{(\Gamma)} \frac{R(\zeta)\,d\zeta}{\zeta - x} = 0. \tag{3.1}$$

The third term on the left tends to zero as $M \to \infty$, the second term to $-\pi i R(x)$ as $\epsilon \to 0$. Thus $\mathfrak{H}R = iR(x)$.

We shall now deal with $\mathfrak{H}f$ in the domain Q, that is to say under the hypothesis $f(t) \in Q$. By (1.1) and Lemma 1, we have

$$f(z) = R(z)+R^*(z), \qquad \mathfrak{H}f = iR(x)-iR^*(x), \tag{3.2}$$

$$\mathfrak{H}^2 f = i\mathfrak{H}(R-R^*) = i^2R(x)+i^2R^*(x) = -f(x). \tag{3.3}$$

Thus in the domain Q the properties (A) and (C) are evident. Now we have

LEMMA 2. *Let $F(z)$ be a rational function, regular for $y \geqslant 0$ or for $y \leqslant 0$, and let $F(z) = O(|z|^{-2})$ as $|z| \to \infty$. Then*

$$\int\limits_{-\infty}^{\infty} F(t)\,dt = 0. \tag{3.4}$$

Let $f_j(z) \in Q, f_j(z) = R_j(z)+R_j^*(z)$ $(j = 1, 2)$. Then, by the lemma,

$$\int\limits_{-\infty}^{\infty} R_1(x)R_2(x)\,dx = \int\limits_{-\infty}^{\infty} R_1^*(x)R_2^*(x)\,dx = 0, \tag{3.5}$$

† S. Banach, *Théorie des opérations linéaires* (Warsaw 1932), 58, Theorem 7.

and so, by (3.2), we arrive at (D). Hence, in the domain Q, $\mathfrak{H}f$ possesses also the property (D), while (E) is self-evident.

Let us now suppose that, for $1 < p < \infty$, (B) holds in the domain Q,

$$|\mathfrak{H}S|_p \leqslant C_p|S|_p, \qquad S(z) \in Q. \tag{3.6}$$

Given an arbitrary element $f(t) \in L_p$, there is a sequence $\{S_n(z)\} \in Q$ such that $|f(t)-S_n(t)|_p \to 0$ as $n \to \infty$. By a well-known argument, from (3.6) we deduce the existence of the limit in mean, of index p, of $\mathfrak{H}S_n$ $(n \to \infty)$. We denote it by $\mathfrak{H}^{(p)}f$. Since it is uniquely determined, it is identical with $\mathfrak{H}f$ in the domain Q. By well-known properties of the limit in mean, the operator $\mathfrak{H}^{(p)}f$ satisfies (A)–(E). It remains to prove (3.6) and to show that $\mathfrak{H}^{(p)}f = \mathfrak{H}f$.

4. We need some lemmas.

LEMMA 3. *Let p be an even positive integer, let $R(z) \in E$, $R^*(z) = E^*$; then*

$$A_p^{-1}|\Re\{R(t)+R^*(t)\}|_p \leqslant |\Im\{R(t)-R^*(t)\}|_p \leqslant A_p|\Re\{R(t)+R^*(t)\}|_p, \tag{4.1}$$

where the constant A_p depends on p only.

Let $f(t) = \Re\{R(t)+R^*(t)\}$, $g(t) = \Im\{R(t)-R^*(t)\}$. Then

$$f(t)+ig(t) = R(t)+\bar{R}^*(t) = R_1(t) \quad \text{where} \quad R_1(z) \in E.$$

By Lemma 2 we have

$$\int_{-\infty}^{\infty} \{f(t)+ig(t)\}^p\, dt = 0.$$

Let A_p be the positive root of $X^p = \binom{p}{2}X^{p-2}+\binom{p}{6}X^{p-6}+\ldots$, then, by an argument due to M. Riesz,† $A_p^{-1}|f|_p \leqslant |g|_p \leqslant A_p|f|_p$, which proves the lemma.

LEMMA 4. *Let $v > 0$, let* $\Phi_v(t) = \begin{cases} t^{-1} & (|t| > v), \\ 0 & (|t| \leqslant v). \end{cases}$
Then, if $1 < p < \infty$,

$$\mathfrak{H}^{(p)}\phi_v = \mathfrak{H}\phi_v = (\pi x)^{-1}\log|(x+v)(x-v)^{-1}|.$$

From the results of § 3 (Lemma 1 and (D)) we deduce that

$$\int_{-\infty}^{\infty} \frac{(i-x)^n}{(i+x)^{n+1}}\mathfrak{H}^{(p)}\phi_v\, dx = -i\Lambda\left\{\int_{-\infty}^{-v} + \int_{v}^{\infty}\right\}\frac{(i-t)^n}{(i+t)^{n+1}}\frac{dt}{t} = U_n,$$

† M. Riesz, loc. cit.; cf. J. Cossar, loc. cit. 379.

where $\Lambda = +1$ for $n \geqslant 0$, $\Lambda = -1$ for $n < 0$. Evaluating $\mathfrak{H}\phi_v$, we have

$$\int_{-\infty}^{\infty} \frac{(i-x)^n}{(i+x)^{n+1}} \mathfrak{H}\phi_v \, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(i-x)^n}{(i+x)^{n+1}} \log\left|\frac{x+v}{x-v}\right| \frac{dx}{x} = V_n.$$

It is not difficult to show that $U_n = U_{-n-1}$, $V_n = V_{-n-1}$, and, taking $v = \tan\frac{1}{2}w$, $-\pi < w < \pi$, that $U_0 = i(w-\pi) = V_0$,

$$iU_n = \pi - w - 2\sum_{k=1}^{n} \frac{\sin kw}{k}, \qquad V_n - V_{n-1} = \frac{2i}{n}\sin nw \quad (n > 0).$$

Consequently $U_n = V_n$ for $n \gtreqless 0$, and so

$$\int_{-\infty}^{\infty} \frac{(i-x)^n}{(i+x)^{n+1}} (\mathfrak{H}^{(p)}\phi_v - \mathfrak{H}\phi_v) \, dx = 0 \quad (n = 0, \pm 1, \pm 2, \ldots).$$

We can now apply the Corollary 1 which shows that $\mathfrak{H}^{(p)}\phi_v = \mathfrak{H}\phi_v$.

5. It will suffice to prove (3.6) under the assumption that p is an even positive integer.† Putting $S = R + R^*$, we have $\mathfrak{H}S = i\,(R - R^*)$. Using the second part of (4.1), also the first part after replacing R^* by $-R^*$, and applying Minkowski's inequality, we arrive at (3.6), with $C_p = 2A_p$. When $p = 2$, then, by Lemma 2,

$$\int \bar{R}R^* \, dt = \int R\bar{R}^* \, dt = 0,$$

and therefore we can take $C_2 = 1$.

Finally, we prove that, for any $f \in L_p$, $\mathfrak{H}^{(p)}f = \mathfrak{H}f$. From (D), (C), (E) and from Lemma 4 we deduce that

$$\frac{1}{\pi}\left\{\int_{-\infty}^{\xi-\epsilon} + \int_{\xi+\epsilon}^{\infty}\right\} \frac{f(t)\,dt}{t-\xi} = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\phi_\epsilon(t-\xi)\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathfrak{H}^{(p)}f\,\mathfrak{H}^{(p)}\{\phi_\epsilon(t-\xi)\}\,dx$$

$$= \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{\mathfrak{H}^{(p)}f}{x-\xi} \log\left|\frac{x-\xi+\epsilon}{x-\xi-\epsilon}\right| dx.$$

By a result due to Hardy,‡ the latter term tends to $\mathfrak{H}^{(p)}f = \mathfrak{H}^{(p)}[f;\xi]$

† By § 3, (3.6) implies

$$(B'): |\mathfrak{H}^{(p)}f|_p \leqslant C_p|f|_p \quad \text{and} \quad (D'): \int S\mathfrak{H}^{(p)}f\,dx = -\int f\mathfrak{H}S\,dx \quad (f \in L_p).$$

By a well-known convexity theorem (M. Riesz, *Acta Math.* 49 (1927), 465–97), (B') holds for $2 \leqslant p < \infty$; by (D'), and by Hölder's inequality and its converse, we have $|\mathfrak{H}S|_{p'} \leqslant C_p|S|_{p'}$ $(1/p + 1/p' = 1)$.

‡ Hardy, loc. cit. Cf. J. Cossar, loc. cit.

for almost all ξ in $(-\infty, \infty)$ as $\epsilon \to 0$. Thus $\mathfrak{H}f$ exists almost everywhere and is equal to $\mathfrak{H}^{(p)}f$.

6. Characteristic functions. We shall prove the following result:

THEOREM 2. *Let* $1 < p < \infty$. (i) *A necessary and sufficient condition that* $F(t)$ *belongs to* $L_p(-\infty, \infty)$ *and that* $\mathfrak{H}F = iF$ *or* $\mathfrak{H}F = -iF$ *is that* $F(t)$ *can be approximated, in the mean of index* p, *by rational functions vanishing at infinity and regular in the closed upper* $(y \geqslant 0)$ *or lower* $(y \leqslant 0)$ *half-plane respectively.*

(ii) *The sequence*

$$\{F_n(t)\} = \{\pi^{-\frac{1}{2}}(t-i)^n(t+i)^{-n-1}\} \quad (n = 0, 1, 2, \ldots \text{ or } n = -1, -2, \ldots)$$

is a closed orthogonal and normal system in the space of the characteristic functions which belong to i *or* $-i$ *respectively.*

Let $F(t) \in L_p$ and $\mathfrak{H}F = iF$. By the Corollary 3, we have $|F(t)-S_n(t)|_p \to 0$ as $n \to \infty$, where $S_n(z)$ is represented in the form (1.1). By (3.2) and (B), we have

$$\begin{aligned} 2|F(t)-R_n(t)|_p &= |iF-iS_n+\mathfrak{H}(F-S_n)|_p \\ &\leqslant |F-S_n|_p + |\mathfrak{H}(F-S_n)|_p \leqslant (1+C_p)|F-S_n|_p \end{aligned}$$

which tends to zero as $n \to \infty$. Hence

$$|F(t)-R_n(t)|_p \to 0 \quad (R_n \in E,\ n \to \infty).$$

The proof of the remaining assertions is evident. It can be shown that the first part of the theorem holds for $p = 1$,† if the approximating rational functions satisfy the additional condition

$$R_n(t),\ R_n^*(t) = O(t^{-2}) \quad (t \to \pm\infty).$$

† For the L_1 theory of $\mathfrak{H}f$ *vide* E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937), 5.9 and 5.14; E. Hille and J. D. Tamarkin, *Fundam. Math.* 25 (1935), 329–52; H. Kober, a forthcoming paper in the *Bull. American Math. Soc.*, 1942.

AN EXTENSION OF HYPERGEOMETRIC FUNCTIONS (I)

By T. W. CHAUNDY (*Oxford*)

[Received 25 December 1942]

1. Introduction

IF in a hypergeometric function ${}_rF_s(a_1,\ldots,a_r;c_1,\ldots,c_s;x)$ one of the upper parameters a is a negative integer $-n$, the series terminates with x^n. If, further, x is 1 (or possibly -1), the function may reduce, under suitable conditions on the remaining parameters, to a product $(A_1)_n(A_2)_n\ldots/(B_1)_n(B_2)_n\ldots$, where $(A)_n \equiv \Gamma(A+n)/\Gamma(A)$. I shall call such a product a *gamma product* and say briefly that such a hypergeometric function is *reducible*. Thus, to take an elementary example,

$${}_2F_1(a,-n;c;1) = (c-a)_n/(c)_n$$

by Gauss's theorem. There is a large corpus of literature due to J. Dougall, F. J. W. Whipple, and others on the subject of these reducible hypergeometric functions of unit argument, and it has been fully documented by W. N. Bailey.*

My argument in what follows is this. If F_n denote a reducible hypergeometric function, the infinite series $\sum F_n x^n/n!$ is merely a hypergeometric series of some order; and so, more generally, is the series $\sum \gamma_n F_n x^n/n!$, where γ_n is a gamma product.† This suggests consideration of such series when F_n is no longer reducible. By hypothesis the reducible F_n satisfies a two-term recurrence-relation of the type

$$F_{n+1} = \gamma_n F_n.$$

When not reducible, F_n will be found to satisfy a many-term recurrence-relation

$$F_{n+1} = \gamma_n F_n + \gamma_n' F_{n-1} + \ldots + \gamma_n^{(r)} F_{n-r}, \tag{1}$$

and consequently the function

$$y = \sum_{n=0}^{\infty} \frac{\bar{\gamma}_n F_n}{n!} x^n$$

satisfies a 'many-term' differential equation of the form

$$f(\delta)y = [xf_1(\delta)+x^2f_2(\delta)+\ldots+x^rf_r(\delta)]y. \tag{2}$$

* (1) *passim.*

† To avoid waste of words I shall use γ_n, γ_n',... as conventional symbols for unspecified gamma products.

If, as here, there are in all $r+1$ terms in the differential equation, I shall say that it is of *rank* r, so that the differential equations of hypergeometric functions

$$f(\delta)y = xg(\delta)y$$

are of rank unity. In general 'rank' gives the number of singularities (other than 0, ∞) of the differential equation. There are, of course, $r+1$ terms in the associated recurrence-relation (1), and we can similarly say that this relation is of rank r. As we shall see later, there are difficulties in extending the notion of 'rank' to the function y itself. Here for the most part I shall be interested in forms F_n that just fail to be reducible, i.e. in which the resulting recurrence-relations and differential equations are of rank two: in particular I shall note certain equations of rank two and order two.

Now the integer n may appear in the hypergeometric function F_n not merely as $-n$, the 'parameter of closure', but also in some or all of the other parameters (upper or lower) in forms $a\pm n$, $c\pm n$, or even (more generally) in forms $a+hn$, $c+hn$, where h is an integer. We get in this way various types of F_n, and the technique for obtaining the appropriate recurrence-relation (1) differs a little for the different types. My procedure here will be to take samples of the different types and, so to speak, 'fix' them by giving their differential equations, in the somewhat Micawberish hope that some light may be thrown on these many-term differential equations. Two points of view are possible: we can seek to determine the differential equation of a given type when all the parameters are as general as possible; or we can impose such conditions on the parameters as will reduce the differential equation in rank or order or both. I shall adopt sometimes the one point of view, sometimes the other. In the second case it is not surprising that we meet Saalschützian and 'well-poised' forms, since it is just these forms that provide F_n reducible to gamma products.

A terminated F_n can be written backwards: for instance, we have without difficulty

$${}_2F_1(a,-n;c;p) = \frac{(a)_n}{(c)_n}(-p)^n\,{}_2F_1\left(1-c-n,-n;1-a-n;\frac{1}{p}\right).$$

Thus, in general, any particular F_n represents a pair of types. Moreover, since in this way the pair of types

$${}_2F_1(a+n,-n;c;p), \qquad {}_2F_1(1-c-n,-n;1-a-2n;1/p)$$

are equivalent, we see that parameters of the form $a \pm hn$, where h is an integer, are naturally admissible, though it will be found that the rank and order of the differential equations rise rapidly with increasing h.

We can also consider F_n which lack the parameter of closure $-n$ and so do not terminate. For example, we have formally

$$ {}_rF_s\left[\begin{matrix} a_1,\dots,a_r; \\ c_1,\dots,c_s \end{matrix} 1+x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_r)_n}{n!\,(c_1)_n \dots (c_s)_n}\, {}_rF_s\left[\begin{matrix} a_1+n,\dots,a_r+n \\ c_1+n,\dots,c_s+n \end{matrix}\Big\} x^n\right]. $$

H. B. C. Darling (6) has considered these and certain allied forms. As a rule we get less simple results from these non-terminating F_n.

From considerations that I have advanced elsewhere* it will generally be found that, as $n \to \infty$, the F_n behave like gamma products and that these extended hypergeometric series have accordingly similar intervals of convergence to those of ordinary hypergeometric series.

2. Some elementary differential equations of rank two

I collect here for ready reference certain differential equations of rank two that are soluble in elementary terms.

The first-order equation

$$ [\delta - px(\delta+a) - qx(\delta+b) + pqx^2(\delta+a+b)]y = 0 \tag{3} $$

has the solution $\quad y = (1-px)^{-a}(1-qx)^{-b}.$

We can write this

$$ y = F^{(1)}[c; a, b; c; px, qx] = \frac{\delta+c-1}{c-1} F^{(1)}[c-1; a, b; c; px, qx]. $$

Thus

$$ [\delta - px(\delta+a) - qx(\delta+b) + pqx^2(\delta+a+b)](\delta+c-1)y = 0 \tag{4} $$

has the solution†

$$ y = F^{(1)}[c-1; a, b; c; px, qx]. \tag{5} $$

By (3) (30) we can write this as

$$ y = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (c-1)_r}{(c+r-1)_r (c)_{2r}} (pqx^2)^r \times $$
$$ \times F(a+r, c-1+r; c+2r; px) F(b+r, c-1+r; c+2r; qx). \tag{6} $$

* (5) § 6.4.

† As here, I usually consider only the series-solution led by a constant term; the other series-solutions are generally to be deduced without difficulty.

Now, when $$e+f+e'+f' = g+h+c-1,$$

$$[\delta(\delta+c-1)-px(\delta+e)(\delta+f)-qx(\delta+e')(\delta+f')+$$
$$+pqx^2(\delta+g)(\delta+h)](1-px)^{c-e-f}y$$
$$= (1-px)^{c-e-f}[\delta(\delta+c-1)-px(\delta+c-f)(\delta+c-e)-$$
$$-qx(\delta+e')(\delta+f')+pqx^2(\delta+g-e-f+c)(\delta+h-e-f+c)]y, \quad (7)$$

and we deduce that

$$[\delta(\delta+c-1)-px(\delta+1)(\delta+c-a)-qx(\delta+b)(\delta+c-1)+$$
$$+pqx^2(\delta+b+1)(\delta+c-a)]y = 0, \quad (8)$$

i.e. $$[\delta-qx(\delta+b)][\delta+c-1-px(\delta+c-a)]y = 0,$$

has the solution

$$y = (1-px)^{a-1}F^{(1)}[c-1;a,b;c;px,qx] \quad (9)$$
$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c-1)_r}{(c+r-1)_r(c)_{2r}}(pqx^2)^r F(c-a+r,1+r;c+2r;px)\times$$
$$\times F(b+r,c-1+r;c+2r;qx); \quad (10)$$

and that

$$[\delta(\delta+c-1)-px(\delta+1)(\delta+c-a)-qx(\delta+1)(\delta+c-b)+$$
$$+pqx^2(\delta+2)(\delta+c-a-b+1)]y = 0, \quad (11)$$

i.e.

$$\delta[\delta+c-1-px(\delta+c-a)-qx(\delta+c-b)+pqx^2(\delta+c-a-b+1)]y = 0,$$

has the solution

$$y = (1-px)^{a-1}(1-qx)^{b-1}F^{(1)}[c-1;a,b;c;px,qx] \quad (12)$$
$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c-1)_r}{(c+r-1)_r(c)_{2r}}(pqx^2)^r F(c-a+r,1+r;c+2r;px)\times$$
$$\times F(c-b+r,1+r;c+2r;qx). \quad (13)$$

Here (10), (13) are deduced from (6) by using the identity

$$(1-x)^{A+B-C}F(A,B;C;x) = F(C-A,C-B;C;x). \quad (14)$$

Now, if $y = \sum F_n x^n$ satisfies a differential equation

$$[f_1(\delta)-xf_2(\delta)+x^2f_3(\delta)]y = 0,$$

so that F_n has the recurrence-relation

$$f_1(n)F_n-f_2(n-1)F_{n-1}+f_3(n-2)F_{n-2} = 0,$$

then $$G_n \equiv (a)_n F_n/(c)_n$$

has the recurrence-relation

$$(c+n-1)(c+n-2)f_1(n)G_n-(a+n-1)(c+n-2)f_2(n-1)G_{n-1}+$$
$$+(a+n-1)(a+n-2)f_3(n-2)G_{n-2} = 0,$$

and so
$$y = \sum \frac{(a)_n}{(c)_n} F_n x^n$$

satisfies the differential equation

$$[(\delta+c-1)(\delta+c-2)f_1(\delta)-x(\delta+a)(\delta+c-1)f_2(\delta)+$$
$$+x^2(\delta+a)(\delta+a+1)f_3(\delta)]y = 0.$$

I shall refer to this procedure as 'augmenting the coefficients' or occasionally, since we may be reversing the process, as 'varying the coefficients'. We should remember that such an 'augmentation', if we use Euler's integral for the beta function, gives an Eulerian transform of the original function. Thus

$$\mathrm{B}(a,c-a)\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}u_n(x) = \int_0^1 t^{a-1}(1-t)^{c-a-1}\Big\{\sum_{n=0}^{\infty}u_n(xt)\Big\}\,dt,$$

with suitable restrictions on a, c.

If in (3) we write b, $c-b$ instead of a, b and augment the coefficients by the factor $(a)_n/(c)_n$, we get, on removal of a common factor, that

$$[\delta(\delta+c-1)-px(\delta+b)(\delta+a)-qx(\delta+c-b)(\delta+a)+$$
$$+pqx^2(\delta+a)(\delta+a+1)]y = 0 \quad (15)$$

has the solution
$$y = F^{(1)}[a;b,c-b;c;px,qx] \quad (16)$$

$$= \sum_{r=0}^{\infty}\frac{(a)_r(b)_r(c-a)_r(c-b)_r}{r!\,(c+r-1)_r(c)_{2r}}(pqx^2)^r F(a+r,b+r;c+2r;px)\times$$
$$\times F(a+r,c-b+r;c+2r;qx), \quad (17)$$

again by (3) (30).

It should be noted that the equations (4), (15) have each a solution of the form
$$y = F^{(1)}[a;b,b';c;px,qx]$$

under the respective conditions (i) $a = c-1$, (ii) $b+b' = c$. Burchnall has shown that the unrestricted $F^{(1)}$ satisfies a third-order equation of rank two.* This equation reduces to the second order by removal of a factor under either of the conditions (i), (ii).

* (2) 92 (9).

Three other second-order equations of rank two are perhaps also in place here. It is readily found that

$$y = (1-qx)^{-1}F(a,b;c;px) \tag{18}$$

satisfies

$$[\delta(\delta+c-1)-px(\delta+a)(\delta+b)-qx(\delta+1)(\delta+c)+ \\ +pqx^2(\delta+a+1)(\delta+b+1)]y = 0, \tag{19}$$

and that

$$y = (1-x)^{-h}F(a,b;c;x) \tag{20}$$

satisfies

$$[\delta(\delta+c-1)-x\{2\delta^2+(a+b+c+2h-1)\delta+ab+ch\}+ \\ +x^2(\delta+a+h)(\delta+b+h)]y = 0. \tag{21}$$

Finally,

$$[\delta(\delta+2c-2)-x^2(\delta+2a)(\delta+2b)]y = 0 \tag{22}$$

is satisfied by

$$y = F(a,b;c;x^2)$$
$$= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(c-a)_r(c-b)_r}{r!\,(c+r-1)_r(c)_{2r}}\, x^{2r}F(a+r,b+r;c+2r;x)\times \\ \times F(a+r,b+r;c+2r;-x), \tag{23}$$

from the duplication formula (4) (24).

In (4), (8), (11), (15), (19) the 'finite' singularities are at $x = 1/p$, $1/q$; in (21) they coincide at $x = 1$; in (22) the finite singularities ± 1 are harmonic conjugates of the singularities 0, ∞.

3. Generalities

3.1. The various types of extended hypergeometric functions can be indicated in terms of those parameters that involve n. Thus I shall describe

$$\sum_{n=0}^{\infty} \frac{\gamma_n}{n!}\, {}_2F_1(a,-n;c;p)x^n, \qquad \sum_{n=0}^{\infty} \frac{\gamma_n}{n!}\, {}_2F_1(a+n,-n;c;p)x^n,$$
$$\sum_{n=0}^{\infty} \frac{\gamma_n}{n!}\, {}_2F_1(a,-n;c+n;p)x^n$$

as of respective types $F(-n)$, $F(a+n,-n)$, $F(-n;c+n)$, and so on.

3.2. Confluent forms can occur in two ways: the terminated hypergeometric series F_n may itself be of confluent type ${}_rF_s$ $(r < s+1)$, or the products indicated by γ_n may be 'confluent' in the sense that there is an excess of products in the denominator. Since such con-

fluent forms are deducible by limiting processes from the more general forms, I shall not go out of my way to obtain them.

3.3. Among the F_n we can include double (or multiple) hypergeometric series, such as

$$F^{(2)}[-n;b,b';c,c';p,q], \qquad F^{(1)}[-n;b,b';c+n;p,q],$$

so long as n appears only in 'double' parameters. More generally we can construct $F_{m,n}$ to satisfy double recurrence-relations, the corresponding extended hypergeometric functions satisfying partial differential equations. Here the $F_{m,n}$ can be constructed out of simple or double terminating functions; in the latter case m, n appear in the single parameters. Thus, as possible $F_{m,n}$ we could consider

$$\text{(i)}\ {}_2F_1(-m,-n;c;p), \qquad \text{(ii)}\ F^{(3)}[-m,-n;b,b';c;p,q].$$

For reasons of space I defer consideration of these possibilities till a later communication.

4. A useful transformation

A useful change of form can be given to these extended hypergeometric functions. In effect, we express them as repeated sums by expanding the terminated hypergeometric function and then re-sum interchanging the order of summation. The procedure is given compactly in terms of differential operators. I note first that for a hypergeometric function F of any order*

$$\frac{(\delta+h)_m}{(\delta+k)_m}F\begin{bmatrix}a, & ..., & ; \\ c, & ..., & ;\end{bmatrix} x = \frac{(h)_m}{(k)_m}F\begin{bmatrix}a, & ..., & h+m, & k; \\ c, & ..., & h, & k+m;\end{bmatrix} x,$$

as we see on comparing the coefficients of x^n. Again

$$\frac{(1-h-\delta)_m(-\delta)_m}{(1-k-\delta)_m}F\begin{bmatrix}a, & ...; \\ c, & ...;\end{bmatrix} x$$

$$= \frac{(a)_m...(1-h-\delta)_m}{(c)_m...(1-k-\delta)_m}(-x)^m F\begin{bmatrix}a+m, & ...; \\ c+m, & ...;\end{bmatrix} x$$

$$= \frac{(a)_m...}{(c)_m...}(-x)^m \frac{(\delta+h)_m}{(\delta+k)_m}F\begin{bmatrix}a+m, & ...; \\ c+m, & ...;\end{bmatrix} x$$

$$= \frac{(a)_m...(h)_m}{(c)_m...(k)_m}(-x)^m F\begin{bmatrix}a+m, & ..., & h+m, & k; \\ c+m, & ..., & h, & k+m;\end{bmatrix} x.$$

* For the moment I omit the order-suffixes r, s.

Then

$$\sum_{n=0}^{\infty} \frac{(a)_n \ldots}{n!\,(c)_n \ldots} F\begin{bmatrix} n+h, \ 1-h'-n, \ -n, \ A, \ \ldots; \\ n+k, \ 1-k'-n, \ C, \ \ldots; \end{bmatrix} p \Bigg] x^n$$

$$= F\begin{bmatrix} \delta+h, \ 1-h'-\delta, \ -\delta, \ A, \ \ldots; \\ \delta+k, \ 1-k'-\delta, \ C; \end{bmatrix} p \Bigg] F\begin{bmatrix} a, \ \ldots; \\ c, \ \ldots; \end{bmatrix} x \Bigg]$$

$$= \sum_{m=0}^{\infty} p^m \frac{(A)_m \ldots (1-h'-\delta)_m (-\delta)_m (\delta+h)_m}{m!\,(C)_m \ldots (1-k'-\delta)_m (\delta+k)_m} F\begin{bmatrix} a, \ \ldots; \\ c, \ \ldots; \end{bmatrix} x \Bigg]$$

$$= \sum_{m=0}^{\infty} \frac{(A)_m \ldots (a)_m \ldots (h+m)_m (h')_m}{m!\,(C)_m \ldots (c)_m \ldots (k+m)_m (k')_m} (-px)^m \times$$

$$\times F\begin{bmatrix} a+m, \ \ldots, \ h+2m, \ k+m, \ h'+m, \ k'; \\ c+m, \ \ldots, \ h+m, \ k+2m, \ h', \ k'+m; \end{bmatrix} x \Bigg]. \qquad (24)$$

This gives the result of the transformation for a general type $F(h+n, 1-h'-n, -n; k+n, 1-k'-n)$ and is sufficient to show the effect of the transformation on more elaborate or simpler types. I note in particular three simple forms:

(i) $$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} F\begin{bmatrix} -n, \ A, \ \ldots; \\ C, \ \ldots; \end{bmatrix} p \Bigg] x^n$$

$$= \sum_{m=0}^{\infty} \frac{(A)_m \ldots (a)_m}{m!\,(C)_m \ldots} (-px)^m (1-x)^{-(a+m)}$$

$$= (1-x)^{-a} F\begin{bmatrix} a, \ A, \ \ldots; \\ C, \ \ldots; \end{bmatrix} -\frac{px}{1-x} \Bigg]; \qquad (25)$$

(ii) $$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} F\begin{bmatrix} -n, \ a+n, \ A, \ \ldots; \\ C, \ \ldots; \end{bmatrix} p \Bigg] x^n$$

$$= \sum_{m=0}^{\infty} \frac{(A)_m \ldots (a)_m (a+m)_m}{m!\,(C)_m \ldots} (-px)^m \ {}_2F_1\begin{bmatrix} a+m, \ a+2m; \\ a+m; \end{bmatrix} x \Bigg]$$

$$= (1-x)^{-a} F\begin{bmatrix} \tfrac{1}{2}a, \ \tfrac{1}{2}a+\tfrac{1}{2}, \ A, \ \ldots; \\ C, \ \ldots; \end{bmatrix} -\frac{4px}{(1-x)^2} \Bigg]; \qquad (26)$$

(iii) $$\sum_{n=0}^{\infty} \frac{(a)_n}{n!} F\begin{bmatrix} -n, \ A, \ \ldots; \\ 1-a-n, \ C, \ \ldots; \end{bmatrix} p \Bigg] x^n$$

$$= (1-x)^{-a} F\begin{bmatrix} A, \ \ldots; \\ C, \ \ldots; \end{bmatrix} px \Bigg]. \qquad (27)$$

The last result shows that the product of a hypergeometric function and a power of $1-x$ is an extended hypergeometric function.

In what follows I shall often be concerned with examples in which γ_n is of the second order, i.e. of the form $(a)_n(b)_n/n!\,(c)_n$. The trans-

formed expression is then a series in ${}_2F_1$. If we bear in mind the latter half of (3) § 1, i.e. such results as (15)–(19) there, we shall see that, for the respective types

$$\text{(i) } F(-n), \qquad \text{(ii) } F(-n;1-a-n), \qquad \text{(iii) } F(-n;c+n),$$

these series proceed in terms of

$$\text{(i) } (-x)^r\, {}_2F_1(a+r,b+r;c+r;x), \qquad \text{(ii) } x^r\, {}_2F_1(a,b+r;c+r;x),$$
$$\text{(iii) } (-x)^r\, {}_2F_1(a+r,b+r;c+2r;x).$$

Five of the series given in § 2 above can be regarded, in this way, as double functions of type $F(-n;c+n)$. They are the series (6), (10), (13), (17), (23), which can be written as

$$[6] \quad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(c-1)_m(a)_m(c-1)_n(b)_n}{m!\,n!\,(c)_m(c)_n}\times$$
$$\times\, {}_4F_3\left[\begin{matrix}1, & \tfrac{1}{2}c+\tfrac{1}{2}, & -m, & -n\\ \tfrac{1}{2}c-\tfrac{1}{2}, & c+m, & c+n\end{matrix}\right](px)^m(qx)^n, \tag{28}$$

$$[10] \quad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(c-a)_m(c-1)_n(b)_n}{n!\,(c)_m(c)_n}\times$$
$$\times\, {}_5F_4\left[\begin{matrix}c-1, & \tfrac{1}{2}c+\tfrac{1}{2}, & a, & -m, & -n\\ \tfrac{1}{2}c-\tfrac{1}{2}, & c-a, & c+m, & c+n\end{matrix}\right](px)^m(qx)^n, \tag{29}$$

$$[13] \quad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(c-a)_m(c-b)_n}{(c)_m(c)_n}\times$$
$$\times\, {}_7F_6\left[\begin{matrix}c-1, & c-1, & \tfrac{1}{2}c+\tfrac{1}{2}, & a, & b, & -m, & -n\\ 1, & \tfrac{1}{2}c-\tfrac{1}{2}, & c-a, & c-b, & c+m, & c+n\end{matrix}\right](px)^m(qx)^n, \tag{30}$$

$$[17] \quad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(b)_m(a)_n(c-b)_n}{m!\,n!\,(c)_m(c)_n}\times$$
$$\times\, {}_5F_4\left[\begin{matrix}c-1, & \tfrac{1}{2}c+\tfrac{1}{2}, & c-a, & -m, & -n\\ & \tfrac{1}{2}c-\tfrac{1}{2}, & a, & c+m, & c+n\end{matrix}\right](px)^m(qx)^n, \tag{31}$$

$$[23] \quad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (-)^n \frac{(a)_m(b)_m(a)_n(b)_n}{m!\,n!\,(c)_m(c)_n}\times$$
$$\times\, {}_6F_5\left[\begin{matrix}c-1, & \tfrac{1}{2}c+\tfrac{1}{2}, & c-a, & c-b, & -m, & -n; \\ & \tfrac{1}{2}c-\tfrac{1}{2}, & a, & b, & c+m, & c+n;\end{matrix}\;-1\right]x^{m+n}, \tag{32}$$

where it may be noted that in (29), (30) factorials in the denominator have cancelled with factorials in the numerator. More significantly, it should be remarked that the terminated hypergeometric series in

the coefficients of the expansions (29) to (32) are all 'well-poised'; in (28) the ${}_4F_3$ is 'nearly well-poised'.

We could apply the transformation of this section to the thirty expansions of (**3**), §§ 3, 4, but there we should find that the eighteen expansions of § 3 are all purely hypergeometric (as indeed they profess to be), for it is of the essence of the lemmas in the preceding § 2 that the five terminated hypergeometric series that occur are all reducible, i.e. of rank one. On the other hand, the six *direct* expansions of § 4 there (i.e. those with even numberings) show that ${}_2F_1(a,b;c;x+y-xy)$, $F^{(4)}[a,b;c,c';x-xy,y-xy]$ can be variously written as extended double hypergeometric functions. Quoting most expeditiously from the lemmas themselves we have from (**3**) (56), (63), (64), (65) respectively that

$$F(a,b;c;x+y-xy)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{m!\,n!\,(c)_{m+n}}\, {}_3F_2\left[\begin{matrix} 1-c-m-n, & -m, & -n \\ 1-a-m-n, & 1-b-m-n \end{matrix}\right] x^m y^n \tag{33}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b)_n}{m!\,n!\,(c)_{m+n}}\, {}_3F_2\left[\begin{matrix} c-a, & -m, & -n \\ b, & 1-a-m-n \end{matrix}\right] x^m y^n \tag{34}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(b)_m(a)_n(b)_n}{m!\,n!\,(c)_{m+n}}\, {}_3F_2\left[\begin{matrix} a+b-c, & -m, & -n \\ a, & b \end{matrix}\right] x^m y^n \tag{35}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(b)_m(a)_n(b)_n}{m!\,n!\,(c)_m(c)_n} \times$$

$$\times\, {}_6F_5\left[\begin{matrix} c-1, & \tfrac{1}{2}c+\tfrac{1}{2}, & c-a, & c-b, & -m, & -n; \\ & \tfrac{1}{2}c-\tfrac{1}{2}, & a, & b, & c+m, & c+n; \end{matrix}\; -1\right] x^m y^n \tag{36}$$

Here (36) is the fuller form of (32) above. Again, from (58), (66) (with a, b interchanged), and the equation following (66) in (**3**), we get

$$F^{(4)}[a,b;c,c';x-xy,y-xy]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{m!\,n!\,(c)_m(c')_n}\, {}_3F_2\left[\begin{matrix} 2-c-c'-m-n, & -m, & -n \\ 1-a-m-n, & 1-b-m-n \end{matrix}\right] x^m y^n \tag{37}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(a)_n(b)_{m+n}}{m!\,n!\,(c)_m(c')_n}\, {}_3F_2\left[\begin{matrix} c+c'-b-1, & -m, & -n \\ a, & 1-b-m-n \end{matrix}\right] x^m y^n \tag{38}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(b)_m(a)_n(b)_n}{m!\,n!\,(c)_m(c')_n}\, {}_3F_2\left[\begin{matrix} a+b-c-c'+1, & -m, & -n \\ a, & b \end{matrix}\right] x^m y^n. \tag{39}$$

We note in passing that here F and $F^{(4)}$ have alternative representations in extended hypergeometric series. These arise immediately from the standard identities connecting ${}_3F_2$ that formed the basis of the lemmas quoted.

5. The linear type $F(-n)$

I begin with three types which, by the principles of the foregoing section, can be reduced to a hypergeometric form by augmentation of coefficients (if necessary). They are therefore, at the worst, Eulerian transforms of hypergeometric functions, and, though for this reason they have less intrinsic interest, they are useful in explaining the technique, since they enable us to check our results independently. I call these types 'linear' for reasons I presently explain. The first of these types is $F(-n)$.

The simplest examples of this type are

$$\text{(i)}\ {}_2F_1(a,-n;c;p), \qquad \text{(ii)}\ {}_3F_2\begin{bmatrix} a_1, & a_2, & -n \\ & c_1, & c_2 \end{bmatrix}.$$

These are just not reducible and lead, as we should expect, to recurrence-relations of rank two. For (i) write

$$F_n \equiv \frac{(c)_n}{n!}\,{}_2F_1(a,-n;c;px).$$

Then

$$\left.\begin{aligned} (\delta-n)F_n &= -(c+n-1)F_{n-1} \\ (\delta-n)(\delta-n+1)F_n &= (c+n-1)(c+n-2)F_{n-2} \end{aligned}\right\}. \tag{40}$$

Moreover, F_n satisfies the differential equation

$$[\delta(\delta+c-1)-px(\delta+a)(\delta-n)]F_n = 0.$$

Now put $x = 1$ after all differentiations have been performed.* Then

$$[\delta(\delta+c-1)-p(\delta+a)(\delta-n)]F_n = 0. \tag{41}$$

If we can write this as

$$[A_n(\delta-n)(\delta-n+1)-B_n(\delta-n)+C_n]F_n = 0, \tag{42}$$

we have, by (40),

$$A_n(n+c-1)(n+c-2)F_{n-2}-B_n(n+c-1)F_{n-1}+C_n F_n = 0, \tag{43}$$

the expected recurrence-relation of rank two. Identification of (41) and (42) at once gives

$$A_n = 1-p, \quad C_n = n(n+c-1), \quad B_n = 2n+c-2-p(n+a-1),$$

* Or we can omit x and think of δ as $p\partial/\partial p$.

and so the recurrence-relation, after the removal of the factor $n+c-1$, is

$$nF_n-\{2n+c-2-p(n+a-1)\}F_{n-1}+(1-p)(n+c-2)F_{n-2} = 0. \quad (44)$$

Thus
$$y = \sum_{n=0}^{\infty} \frac{(c)_n}{n!}\, {}_2F_1(a, -n; c; p)x^n \quad (45)$$

satisfies the differential equation

$$[\delta-x\{2\delta+c-p(\delta+a)\}+x^2(1-p)(\delta+c)]y = 0. \quad (46)$$

The coefficient $(c)_n/n!$ in (45) is of the first order and, by the principles of § 4 above, we can express y in the elementary form

$$y = (1-x)^{a-c}\{1-(1-p)x\}^{-a}, \quad (47)$$

which is otherwise known to be the solution of the equation (46).

Similarly with (ii), writing

$$F_n \equiv \frac{(c)_n}{n!}\, {}_3F_2\left[\begin{matrix} a, & a', & -n \\ & c, & c' \end{matrix}\right],$$

we have the same recurrence-formulae (40). When we put $x = 1$ in the third-order differential equation for F_n, it reduces to the second-order form

$$[\delta(\delta+c-1)(\delta+c'-1)-(\delta+a)(\delta+a')(\delta-n)]F_n = 0$$

by cancellation of δ^3. We can therefore again assume a form (42) leading to a recurrence-relation (43). Evaluating the coefficients, we get in this case

$$\begin{aligned} & n(n+c'-1)F_n- \\ & \quad -\{2(n-1)^2+(2c+2c'-a-a'-1)(n-1)+cc'-aa'\}F_{n-1}+ \\ & \qquad +(n+c-2)(n+c+c'-a-a'-2)F_{n-2} = 0. \end{aligned}$$

Thus,
$$y = \sum_{n=0}^{\infty} \frac{(c)_n}{n!}\, {}_3F_2\left[\begin{matrix} a, & a', & -n \\ & c, & c' \end{matrix}\right]x^n \quad (48)$$

satisfies the differential equation

$$\begin{aligned} & [\delta(\delta+c'-1)-x\{2\delta^2+(2c+2c'-a-a'-1)\delta+cc'-aa'\}+ \\ & \qquad +x^2(\delta+c)(\delta+c+c'-a-a')]y = 0. \quad (49) \end{aligned}$$

Here again, by (25), we can write

$$\begin{aligned} y &= (1-x)^{-c}\, {}_2F_1\left(a, a'; c'; -\frac{x}{1-x}\right) \\ &= (1-x)^{a-c}\, {}_2F_1(a, c'-a'; c'; x) \quad (50) \end{aligned}$$

by a familiar identity. Alternatively, we can identify (49) with the

equation (21) above by writing a, $c'-a'$, c', $c-a$ for a, b, c, h. This identifies (50) with the solution (20). But by (27) we can write (50) in the form

$$\sum_{n=0}^{\infty} \frac{(c-a)_n}{n!} \, {}_3F_2\begin{bmatrix} a, & c'-a', & -n \\ c', & 1+a-c-n & \end{bmatrix} x^n.$$

Comparing this with (48) we recover a known identity in ${}_3F_2$, obtaining a further instance of the equivalence of two types of extended hypergeometric series.*

We can achieve less elementary results by augmentation of coefficients as explained in § 2. Thus by augmentation of (48) we can consider

$$y = \sum_{n=0}^{\infty} \frac{(c)_n (c')_n}{n!\,(k)_m} \, {}_3F_2\begin{bmatrix} a, & a', & -n \\ & c, & c' \end{bmatrix} x^n, \tag{51}$$

which, by (24), can be written

$$y = \sum_{m=0}^{\infty} (-)^m \frac{(a)_m (a')_m}{m!\,(k)_n} x^m \, {}_2F_1(c+m, c'+m; k+m; x). \tag{52}$$

We find that it satisfies the third-order equation

$$\begin{aligned} [\delta(\delta+k-1)(\delta+k-2)- \\ -x(\delta+k-1)\{2\delta^2+(2c+2c'-a-a'-1)\delta+cc'-aa'\}+ \\ +x^2(\delta+c)(\delta+c')(\delta+c+c'-a-a')]y = 0. \end{aligned} \tag{53}$$

Here we may note that, if we write $k = c+c'-a-a'$, the differential equation reduces, after removal of a left-hand factor, to

$$\begin{aligned} [\delta(\delta+c+c'-a-a'-1)-x\{2\delta^2+(2c+2c'-a-a'-1)\delta+cc'-aa'\}+ \\ +x^2(\delta+c)(\delta+c')]y = 0, \end{aligned} \tag{54}$$

which we identify with (21) if we write c, c', $c+c'-a-a'$ for $a+h$, $b+h$, c and a or a' for h. Thus, from (20),

$$y = (1-x)^{-a} \, {}_2F_1(c-a, c'-a; c+c'-a-a'; x). \tag{55}$$

By (51) and (52),

$$y = \sum_{n=0}^{\infty} \frac{(c)_n (c')_n}{n!\,(c+c'-a-a')_n} \, {}_3F_2\begin{bmatrix} a, & a', & -n \\ & c, & c' \end{bmatrix} x^n \tag{56}$$

$$\begin{aligned} = \sum_{m=0}^{\infty} (-)^m \frac{(a)_m (a')_m}{m!\,(c+c'-a-a')_m} x^m \times \\ \times {}_2F_1(c+m, c'+m; c+c'-a-a'+m; x). \end{aligned} \tag{57}$$

* The identity is a particular case of a formula given by G. H. Hardy (7) (5.2); see also Bailey (1) 21, § 3.8 (1).

I call the type $F(-n)$ 'linear' because, in (40), F_n, F_{n-1} are connected by a linear operation in δ. The same is true of the two types now to be discussed.

6. The linear types $F(-n;c-n)$ and $F(a+n,-n)$

For the type $F(-n;c-n)$ we note that

$$ {}_2F_1(a,-n;1-c-n;p) = \frac{(c+a)_n}{(c)_n}\,{}_2F_1(a,-n;c+a;1-p), $$

$$ {}_3F_2\begin{bmatrix} a, & a', & -n \\ c', & 1-c-n \end{bmatrix} = \frac{(c+a)_n}{(c)_n}\,{}_3F_2\begin{bmatrix} a, & c'-a', & -n \\ & c', & c+a \end{bmatrix}, $$

so that to this order we get nothing new.* Now write

$$ F_n \equiv \frac{(2-c)_n}{n!}\,{}_4F_3\begin{bmatrix} a_1, & a_2, & a_3, & -n \\ c_1+1, & c_2+1, & c-n-1 \end{bmatrix}, $$

$$ F \equiv \frac{(2-c)_{n-2}}{n!}\,{}_4F_3\begin{bmatrix} a_1, & a_2, & a_3, & -n \\ c_1+1, & c_2+1, & c-n+1 \end{bmatrix}. $$

Then

$$ F_n = (\delta+c-n)(\delta+c-n-1)F, \qquad F_{n-1} = (\delta+c-n)(\delta-n)F, $$
$$ F_{n-2} = (\delta-n)(\delta-n+1)F. $$

The equation satisfied by F is

$$ [(\delta+c-n)f(\delta)-(\delta-n)g(\delta)]F = 0, $$

where

$$ f(\delta) \equiv \delta(\delta+c_1)(\delta+c_2), \qquad g(\delta) \equiv (\delta+a_1)(\delta+a_2)(\delta+a_3). $$

If $$ c+c_1+c_2 = a_1+a_2+a_3 $$

(so that the ${}_4F_3$ in F_n is Saalschützian), this is quadratic in δ, and we can assume as an equivalent form

$$ [A(\delta+c-n)(\delta+c-n-1)- $$
$$ -B(\delta+c-n)(\delta-n)+C(\delta-n)(\delta-n+1)]F = 0. $$

Evaluation of the constants A, B, C gives the recurrence-relation for F_n, and we deduce that, when the ${}_4F_3$ is Saalschützian,

$$ y = \sum_{n=0}^{\infty} \frac{(2-c)_n}{n!}\,{}_4F_3\begin{bmatrix} a_1, & a_2, & a_3, & -n \\ c_1+1, & c_2+1, & c-n-1 \end{bmatrix} x^n \tag{58} $$

satisfies the third-order equation

$$ [f(\delta)-x\Delta+x^2g(\delta-c+2)]y = 0, \tag{59} $$

* The identity in ${}_3F_2$ is that just noted as a particular case of a formula of Hardy's.

where Δ can be written at will in either of the forms*

$$g(\delta)+(c-1)f(\delta)-(c-2)f(\delta+1),$$

$$f(\delta-c+2)+(c-1)g(\delta-c+2)-(c-2)g(\delta-c+1).$$

From (27) we see that

$$y = (1-x)^{c-2}\,{}_3F_2\left[\begin{matrix} a_1, & a_2, & a_3; \\ c_1+1, & c_2+1 & \end{matrix}\; x\right], \tag{60}$$

which we can use to confirm the form of (59).

We handle the type $F(a+n, -n)$ similarly and find, for example, that

$$y = \sum_{n=0}^{\infty} \frac{(a)_n}{n!}\,{}_2F_1(a+n, -n; c; p)x^n \tag{61}$$

$$= (1-x)^{-a}\,{}_2F_1\left(\tfrac{1}{2}a, \tfrac{1}{2}a+\tfrac{1}{2}; c; -\frac{4px}{(1-x)^2}\right) \tag{62}$$

satisfies the third-order equation

$$[(2\delta+a-3)(\delta+c-1)-x(2\delta+a+1)(\delta+a-c)][\delta-x(\delta+a)]y+$$
$$+px(2\delta+a-1)(2\delta+a)(2\delta+a+1)y = 0; \tag{63}$$

and that

$$y = \sum_{n=0}^{\infty} \frac{(a)_n}{n!}\,{}_3F_2\left[\begin{matrix} a+n, & b, & -n \\ & c_1, & c_2 \end{matrix}\right]x^n \tag{64}$$

$$= (1-x)^{-a}\,{}_3F_2\left[\begin{matrix} \tfrac{1}{2}a, & \tfrac{1}{2}a+\tfrac{1}{2}, & b; \\ & c_1, & c_2 \end{matrix}\; -\frac{4x}{(1-x)^2}\right] \tag{65}$$

satisfies the fourth-order equation

$$[(2\delta+a-3)(\delta+c_1-1)(\delta+c_2-1)+x(2\delta+a+1)(\delta+a-c_1)(\delta+a-c_2)]\times$$
$$\times[\delta-x(\delta+a)]y+bx(2\delta+a-1)(2\delta+a)(2\delta+a+1)y = 0. \tag{66}$$

As some check on these two equations we can put $p = 0$, $b = 0$ respectively, which gives in both cases $y = (1-x)^{-a}$, i.e.

$$[\delta-x(\delta+a)]y = 0.$$

In virtue of the identity

$$(2\delta+a-1)[(\delta+1)(\delta+a-c+1)-(\delta+a)(\delta+c)]+$$
$$+(2\delta+a+1)[(\delta+a-1)(\delta+c-1)-\delta(\delta+a-c)] = 0,$$

we can write (63) in the alternative form

$$[(2\delta+a-3)\delta-x(2\delta+a+1)(\delta+a-1)][(\delta+c-1)-x(\delta+a-c+1)]y+$$
$$+px(2\delta+a-1)(2\delta+a)(2\delta+a+1)y = 0, \tag{67}$$

* These forms are obtainable by setting in turn $\delta = n-1$, $\delta = n-c+1$ to evaluate B; they are evidently complementary.

which resumes its original form (63) under the substitution

$$y = x^{1-c}y', \qquad a = a'+2c-2, \qquad c = 2-c'.$$

Thus (63) has the second solution

$$x^{1-c}\sum_{n=0}^{\infty}\frac{(a-2c+2)_n}{n!}\,{}_2F_1(a-2c+n+2,-n;2-c;p)x^n, \tag{68}$$

which evidently corresponds to the indicial factor $\delta+c-1$. Again, if we multiply (63) by $1+x$, we can rewrite it as

$$(2\delta+a-3)[\{(\delta+c-1)-(a-2c+3)x-x^2(\delta+a-c)\}\{\delta-x(\delta+a)\}+$$
$$+\{px(2\delta+a+1)+px^2(2\delta+a-1)\}(2\delta+a)]y = 0. \tag{69}$$

Thus (61), (68) are solutions of the second-order equation of rank three

$$\Delta y \equiv [(\delta+c-1)-(a-2c+3)x-x^2(\delta+a-c)][\delta-x(\delta+a)]y+$$
$$+[px(2\delta+a+1)+px^2(2\delta+a-1)](2\delta+a)y = 0, \tag{70}$$

while the solution of (63) corresponding to the indicial factor $2\delta+a-3$ satisfies the equation

$$\Delta y = x^{-\frac{1}{2}(a-3)}$$

and, by the usual theory, is expressible as an integral in terms of a suitable Green's function derived from (61), (68).

It is to be remarked that, in passing from (63) to (70) as differential equations both satisfied by (61) and (69), we have decreased the order of the equation at the cost of increasing its rank. This phenomenon has been described elsewhere by J. L. Burchnall: in particular with reference to equation (82) in the following section, where I shall have more to say on it. At any rate it is clear that we cannot immediately transfer the idea of 'rank' from the differential equation to the functions satisfying it.

7. The quadratic type $F(a-n,-n)$

As examples of this type we can consider the forms

$${}_0F_1(c;px)\,{}_0F_1(c';qx) = \sum_{n=0}^{\infty}\frac{(px)^n}{n!\,(c)_n}\,{}_2F_1(1-c-n,-n;c';q/p), \tag{71}$$

$${}_1F_1(a;c;px)\,{}_1F_1(a';c';qx)$$
$$= \sum_{n=0}^{\infty}\frac{(a)_n(px)^n}{n!\,(c)_n}\,{}_3F_2\begin{bmatrix}a',\ 1-c-n,\ -n; & -\dfrac{q}{p}\\ c',\ 1-a-n; & \end{bmatrix}, \tag{72}$$

$$ {}_2F_0(a,b;px))\,{}_2F_0(a',b';qx) $$
$$ = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(px)^n}{n!}\,{}_3F_2\left[\begin{matrix} a',\ b',\ -n; \\ 1-a-n,\ 1-b-n; \end{matrix} -\frac{q}{p}\right], \tag{73} $$

$$ {}_2F_1(a,b;c;px)\,{}_2F_1(a',b';c';qx) $$
$$ = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(px)^n}{n!\,(c)_n}\,{}_4F_3\left[\begin{matrix} a',\ b',\ 1-c-n,\ -n; \\ c',\ 1-a-n,\ 1-b-n; \end{matrix} \frac{q}{p}\right], \tag{74} $$

the series on the right being got at once by diagonal summation on the left, i.e. by a reversal of the analysis of § 4. From the first three of these we derive, by augmentation of the coefficients, the analogous series for $F^{(4)}$, $F^{(2)}$, $F^{(3)}$ in arguments px, qx: in fact, this is the chief justification for including the product of the improper series ${}_2F_0$.

The product (71) and the associated $F^{(4)}$ have already been discussed by Burchnall, who found that they satisfy differential equations of rank two.* I begin, then, with (71), writing

$$ F_n = {}_2F_1(1-c-n,-n;c';q/p)/n!\,(c)_n, $$

so that
$$ F_{n-1} = (\delta-n)(\delta+1-c-n)F_n, $$

and consequently the type is quadratic. If F_n is to satisfy a recurrence-relation of rank two, there must be an identity of the form

$$ A(\delta-n)(\delta-n+1)(\delta+1-c-n)(\delta+2-c-n)F_n+ $$
$$ +B(\delta-n)(\delta+1-c-n)F_n+CF_n = 0 \tag{75} $$

deducible from the differential equation satisfied by F_n, namely

$$ p\delta(\delta+c'-1)F_n = q(\delta-n)(\delta+1-c-n)F_n. \tag{76} $$

Since (75) is of the fourth order in δ and (76) only of order two, we must enlarge the second equation by operating with some arbitrary quadratic operator $\lambda(\delta)$. The operation must, of course, be performed before we put $x = 1$, and so we get the quartic

$$ p\delta(\delta+c'-1)\lambda(\delta)F_n = q(\delta-n)(\delta+1-c-n)\lambda(\delta+1)F_n. \tag{77} $$

If we count the arbitrary constants A, B, C together with those in $\lambda(\delta)$, we see that they are just enough to allow identification of corresponding coefficients in (75), (77).

Similar counting will show that the products (72), (73), and consequently $F^{(2)}$, $F^{(3)}$ in arguments px, qx, satisfy equations of rank three, and the product (74) an equation of rank four. The

* (2) (35), (36).

rank of these equations, however, reduces to two, if $p = -q$ in (72), (73); and in (74) if

$$p = q, \qquad a+b+a'+b' = c+c'.$$

For under these conditions the equations analogous to (76) reduce to the second order in δ. The simpler cases of (72), (73) need not be separately considered since they can be deduced by confluence from the corresponding simpler case of (74).

The evaluation of the arbitrary constants in (75) by direct identification of this form with (77) is rather tiresome, and I proceed differently, showing that the differential equation in question is an augmentation of (63) above.

We know of Kummer's twenty-four solutions of the ordinary hypergeometric differential equation

$$\delta(\delta+c-1)F = x(\delta+a)(\delta+b)F$$

that, while some can be identified in pairs, e.g. $F(a,b;c;x)$ and $(1-x)^{c-a-b}F(c-a,c-b;c;x)$, as a rule they are linked, linearly, in sets of three. When, however, the hypergeometric series is terminated, the twenty-four solutions fall into two groups of twelve, one group corresponding to each of the 'indicial' factors δ, $\delta+c-1$; and the twelve solutions in either group are identical after multiplication by some gamma product. For instance, to reverse the order of the terms in the finite series ${}_2F_1(a,-n;c;x)$ is to identify it with a solution in argument x^{-1}. The identity I require here is

$${}_2F_1(a,-n;c;x) = \frac{(a)_n}{(c)_n}(1-x)^n\,{}_2F_1\left(c-a,-n;1-a-n;\frac{1}{1-x}\right),$$

which can, of course, be verified by direct expansion and rearrangement on the right. If we use this in (71) and remember that $(1-c-n)_n = (-)^n(c)_n$, we get the identity

$$\begin{aligned} &{}_0F_1(c;px)\,{}_0F_1(c';qx) \\ &\qquad = \sum_{n=0}^{\infty}\frac{\{-(p-q)x\}^n}{n!\,(c')_n}\,{}_2F_1\left(c+c'+n-1,-n;c;\frac{p}{p-q}\right). \qquad (78)\end{aligned}$$

The form on the right is now of the linear type $F(a+n,-n)$ and, in changed symbols, is deducible from (61) by variation of coefficients. Precisely, if we write

$$y = \sum_{n=0}^{\infty}\frac{(c+c'-1)_n}{n!}\{-(p-q)x\}^n\,{}_2F_1\left(c+c'+n-1,-n;c;\frac{p}{p-q}\right), \qquad (79)$$

we have from (63)

$$[(2\delta+c+c'-4)(\delta+c-1)+ \\ +(p-q)x(2\delta+c+c')(\delta+c'-1)][\delta+(p-q)x(\delta+c+c'-1)]y- \\ -px(2\delta+c+c'-2)(2\delta+c+c'-1)(2\delta+c+c')y = 0, \quad (80)$$

i.e., on expansion of the product and reduction of the term in x,

$$(2\delta+c+c'-4)\delta(\delta+c-1)y-\tfrac{1}{2}x(2\delta+c+c'-1)\times \\ \times[(p+q)(2\delta+c+c')(2\delta+c+c'-2)+(p-q)(c'-c)(c+c'-2)]y+ \\ +(p-q)^2x^2(2\delta+c+c'+2)(\delta+c')(\delta+c+c'-1)y = 0. \quad (81)$$

We obtain (78) from (79) if we reduce the coefficients by the factor $(c')_n(c+c'-1)_n$: the consequential change in the equation (81) is to remove the factor $(\delta+c')(\delta+c+c'-1)$ from the last term and insert it, in the form $(\delta+c'-1)(\delta+c+c'-2)$, in the first term. This gives us the equation

$$(2\delta+c+c'-4)\delta(\delta+c-1)(\delta+c'-1)(\delta+c+c'-2)y- \\ -\tfrac{1}{2}x(2\delta+c+c'-1)[(p+q)(2\delta+c+c')(2\delta+c+c'-2)+ \\ +(p-q)(c'-c)(c+c'-2)]y+(p-q)^2x^2(2\delta+c+c'+2)y = 0, \quad (82)$$

which is Burchnall's form.*

We can apply the augmentation process directly to the form (80), obtaining, as equivalent to (82), the equation

$$[(2\delta+c+c'-4)(\delta+c-1)(\delta+c+c'-2)+(p-q)x(2\delta+c+c')]\times \\ \times[\delta(\delta+c'-1)+(p-q)x]y- \\ -px(2\delta+c+c'-2)(2\delta+c+c'-1)(2\delta+c+c')y = 0. \quad (83)$$

If we multiply (83) by $(c-c')(c+c'-2)+4(p-q)x$, we can remove a factor $2\delta+c+c'-4$ from the right, getting, as an equation still satisfied by (78),

$$(c-c')(c+c'-2)(\delta+c-1)(\delta+c+c'-2)+ \\ +[(p-q)x\{4\delta^2+4(2c+c'-4)\delta+5c^2+4cc'-c'^2-20c-4c'+16\}+ \\ +4(p-q)^2x^2][\delta(\delta+c'-1)+(p-q)x]y- \\ -px[(c-c')(c+c'-2)(2\delta+c+c')+4(p-q)x(2\delta+c+c'-2)]\times \\ \times(2\delta+c+c'-1)y = 0. \quad (84)$$

This is the equation of order decreased to four and rank increased to three referred to by Burchnall: it is equivalent to his equation (2) (40) satisfied by the product $J_\mu(ax)J_\rho(bx)$.

* (2) 100 (35).

If we recall that $F^{(4)}[a, b; c, c'; px, qx]$ is got from ${}_0F_1(c; px)\,{}_0F_1(c'; qx)$ by augmentation in the factor $(a)_n(b)_n$, and if we once more replace a, p, x in (61) by $c+c'-1$, $p/(p-q)$, $(q-p)x$, we get

$$F^{(4)}[c+c'-1, c'; c, c'; px, qx].$$

Equating this to the equivalent form of (62) and writing x, y for px, qx we find the identity

$$\begin{aligned} &F^{(4)}[c+c'-1, c'; c, c'; x, y] \\ &\quad = (1+x-y)^{1-c-c'}\,{}_2F_1\left[\tfrac{1}{2}c+\tfrac{1}{2}c'-\tfrac{1}{2}, \tfrac{1}{2}c+\tfrac{1}{2}c'; c; \frac{4x}{(1+x-y)^2}\right]. \end{aligned} \tag{85}$$

8. The product ${}_2F_1 \times {}_2F_1$

I now obtain the differential equation of rank two satisfied by

$$y = {}_2F_1(a, b; c; x)\,{}_2F_1(a', b'; c'; x)$$

when $a+b+a'+b' = c+c'$, using a modified technique that throws, perhaps, fresh light on Burchnall's phenomenon. Write

$$\left.\begin{aligned} F_n &\equiv \frac{(a)_n(b)_n}{n!\,(c)_n}\,{}_4F_3\begin{bmatrix} a', & b', & 1-c-n, & -n \\ c', & 1-a-n, & 1-b-n \end{bmatrix} \\ G_n &\equiv \frac{(a)_{n-1}(b)_n}{(n-1)!\,(c)_n}\,{}_4F_3\begin{bmatrix} a', & b', & 1-c-n, & 1-n \\ c', & 2-a-n, & 1-b-n \end{bmatrix} \end{aligned}\right\}, \tag{86}$$

$$\left.\begin{aligned} F &\equiv \frac{(a)_{n-1}(b)_{n-1}}{n!\,(c)_n}\,{}_4F_3\begin{bmatrix} a', & b', & 1-c-n, & -n \\ c', & 2-a-n, & 2-b-n \end{bmatrix} \\ G &\equiv \frac{(a)_{n-2}(b)_{n-1}}{(n-1)!\,(c)_n}\,{}_4F_3\begin{bmatrix} a', & b', & 1-c-n, & 1-n \\ c', & 3-a-n, & 2-b-n \end{bmatrix} \end{aligned}\right\}. \tag{87}$$

Then

$$F_n = (\delta+1-a-n)(\delta+1-b-n)F, \qquad F_{n-1} = (\delta+1-c-n)(\delta-n)F,$$
$$G_n = (\delta+1-b-n)(\delta-n)F;$$

and again

$$G_n = (\delta+2-a-n)(\delta+1-b-n)G, \quad G_{n-1} = (\delta+1-c-n)(\delta+1-n)G,$$
$$F_{n-1} = (\delta+2-a-n)(\delta+1-c-n)G.$$

By our usual methods we obtain the recurrence-relations

$$\begin{aligned} &n(n+c'-1)F_n-(n+b+a'-1)(n+b+b'-1)F_{n-1} \\ &\qquad\qquad = (a-c)(2n+c+c'-2)G_n, \end{aligned}$$

$$\begin{aligned} &(n+c-1)(n+c+c'-2)G_n-(n+a+a'-2)(n+a+b'-2)G_{n-1} \\ &\qquad\qquad = b(2n+c+c'-3)F_{n-1}. \end{aligned}$$

Thus, writing

$$y = \sum_{n=0}^{\infty} x^n F_n = {}_2F_1(a,b;c;x)\,{}_2F_1(a',b';c';x), \qquad z = \sum_{n=1}^{\infty} x^n G_n, \tag{88}$$

we get the pair of simultaneous equations

$$[\delta(\delta+c'-1)-x(\delta+a'+b)(\delta+b'+b)]y = (a-c)(2\delta+c+c'-2)z, \tag{89}$$

$$[(\delta+c-1)(\delta+c+c'-2)-x(\delta+a+a'-1)(\delta+a+b'-1)]z$$
$$= bx(2\delta+c+c'-1)y. \tag{90}$$

To obtain the differential equation in y we eliminate z between these two equations by operating with $(2\delta+c+c'-2)(2\delta+c+c'-4)$ on (90). This gives

$$[(2\delta+c+c'-4)(\delta+c-1)(\delta+c+c'-2)-$$
$$-x(2\delta+c+c')(\delta+a+a'-1)(\delta+a+b'-1)](2\delta+c+c'-2)z$$
$$= bx(2\delta+c+c')(2\delta+c+c'-1)(2\delta+c+c'-2)y,$$

i.e., by (89),

$$[(2\delta+c+c'-4)(\delta+c-1)(\delta+c+c'-2)-$$
$$-x(2\delta+c+c')(\delta+a+a'-1)(\delta+a+b'-1)]\times$$
$$\times[\delta(\delta+c'-1)-x(\delta+a'+b)(\delta+b'+b)]y$$
$$= (a-c)bx(2\delta+c+c')(2\delta+c+c'-1)(2\delta+c+c'-2)y. \tag{91}$$

Confluence readily gives (82), the equation satisfied by the product ${}_0F_1(c;px)\,{}_0F_1(c';qx)$.

More generally, by the confluence $(b,b',x)\to(1/\epsilon,-1/\epsilon,\epsilon x)$ we get the equation satisfied by

$$y_1 = {}_1F_1(a;c;x)\,{}_1F_1(a';c';-x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{n!\,(c)_n}\,{}_3F_2\left[\begin{matrix} a', & 1-c-n, & -n \\ & c', & 1-a-n \end{matrix}\right].$$

If now, remembering that $a+a'+b+b' = c+c'$, we further write

$$a+a' = b+b' \quad (=\tfrac{1}{2}c+\tfrac{1}{2}c'),$$

we can remove an operator $2\delta+c+c'-4$ on the right and rearrange the equation into the form

$$\delta(\delta+c-1)(\delta+c'-1)(\delta+c+c'-2)y_1-$$
$$-(2a-c)x(\delta+a+a')(2\delta+c+c'-1)y_1-$$
$$-x^2(\delta+a+a')(\delta+a+a'+1)y_1 = 0. \tag{92}$$

From this we derive, by augmentation of coefficients, the two equations:

$$[\delta(\delta+c+c'-2)-(2a-c)x(2\delta+c+c'-1)-x^2(\delta+c)(\delta+c')]y_2 = 0 \quad (93)$$

satisfied by

$$y_2 = \sum_{n=0}^{\infty} \frac{(a)_n (c')_n x^n}{n!\,(\frac{1}{2}c+\frac{1}{2}c')_n}\, {}_3F_2\begin{bmatrix} \frac{1}{2}c+\frac{1}{2}c'-a,\ 1-c-n,\ -n \\ c',\ 1-a-n \end{bmatrix}, \quad (94)$$

and

$$[\delta(\delta+c'-1)-(2a-c)x(2\delta+c+c'-1)-x^2(\delta+c)(\delta+c+c'-1)]y_3 = 0 \quad (95)$$

satisfied by

$$y_3 = \sum_{n=0}^{\infty} \frac{(a)_n (c+c'-1)_n x^n}{n!\,(\frac{1}{2}c+\frac{1}{2}c')_n}\, {}_3F_2\begin{bmatrix} \frac{1}{2}c+\frac{1}{2}c'-a,\ 1-c-n,\ -n \\ c',\ 1-a-n \end{bmatrix}. \quad (96)$$

In the notation I have used elsewhere* we can express y_2, y_3 as the double hypergeometric functions

$$\left.\begin{aligned} y_2 &= F\begin{bmatrix} c,\ c':\ a;\ a'; \\ \frac{1}{2}c+\frac{1}{2}c':\ c;\ c'; \end{bmatrix} x,\ -x \Bigg] \\ y_3 &= F\begin{bmatrix} c+c'-1,\ c':\ a;\ a'; \\ \frac{1}{2}c+\frac{1}{2}c':\ \ c;\ c'; \end{bmatrix} x,\ -x \Bigg] \end{aligned}\right\}, \quad (97)$$

where, of course, $a+a' = \frac{1}{2}c+\frac{1}{2}c'$. The equations (93), (95) are of rank and order two. As in the equation (22) above, their 'finite' singularities ± 1 are harmonic conjugates with the singularities $0, \infty$.

The equation (91) is of rank two and order five, whereas the eliminant, as usually understood, of (89) and (90) is of order four. We recover this usual eliminant from (91) if we multiply through by

$$(c-c')(c+c'-2)-(2a+2a'-c-c')(2a+2b'-c-c')x,$$

which enables us to remove an operator $2\delta+c+c'-4$ from the left. The equation has now been reduced to the proper order four at the expense of increasing its rank by one.

9. The eliminant-of-least-rank

This aspect of Burchnall's phenomenon can be described in general terms from the point of view of elimination. Suppose that we are given a pair of differential equations with polynomial coefficients, say

$$\sum_{r=0}^{m}\sum_{s=0}^{n} A_{rs}x^r\delta^s z = \theta, \qquad \sum_{r=0}^{p}\sum_{s=0}^{q} B_{rs}x^r\delta^s z = \phi, \quad (98)$$

* (5) § 3, 163.

where A_{rs}, B_{rs} are constants and θ, ϕ are independent of z. If we are primarily interested in rank, we arrange them in a form

$$\sum_{r=0}^{m} x^r f_r(\delta)z = \theta, \qquad \sum_{r=0}^{p} x^r g_r(\delta)z = \phi$$

and apply Sylvester's 'dialytic' technique by multiplying by $1, x,..., x^p$ and $1, x,..., x^m$ respectively, working these multipliers to the right past the operators $f(\delta)$, $g(\delta)$. We have then $m+p+2$ simultaneous equations in the $m+p+1$ unknowns z, $xz,..., x^{m+p}z$. They are differential equations in δ with constant coefficients, and so the $m+p+1$ unknowns can be eliminated by suitable operations in δ. In fact, if, for simplicity, we take $m = 1$, $p = 2$, we can exhibit the eliminant as

$$\begin{vmatrix} f_0(\delta) & f_1(\delta-1) & 0 & 0 & \theta \\ 0 & f_0(\delta-1) & f_1(\delta-2) & 0 & x\theta \\ 0 & 0 & f_0(\delta-2) & f_1(\delta-3) & x^2\theta \\ g_0(\delta) & g_1(\delta-1) & g_2(\delta-2) & 0 & \phi \\ 0 & g_0(\delta-1) & g_1(\delta-2) & g_2(\delta-3) & x\phi \end{vmatrix} = 0, \tag{99}$$

where the last column is a column of operands on which operate the δ-operators of the other columns. This eliminant may be called the *eliminant-of-least-rank*.

By contrast, the eliminant as ordinarily understood may be described as the *eliminant-of-least-order*. To obtain this we follow the usual practice: we rearrange the equations (98) in a form

$$\sum_{s=0}^{n} h_s(x)\delta^s z = \theta, \qquad \sum_{s=0}^{q} k_s(x)\delta^s z = \phi$$

and apply Sylvester's 'dialytic' technique now in terms of δ, operating with $1, \delta,..., \delta^q$ and $1, \delta,..., \delta^n$ respectively on the two equations and working the operators to the right past the algebraic coefficients. This gives us $n+p+2$ (algebraic) equations linear in the $n+p+1$ unknowns z, $\delta z,..., \delta^{n+p}z$, and the elimination proceeds normally.

To obtain the differential equation of the unrestricted ${}_2F_1 \times {}_2F_1$, i.e. of

$$y = {}_2F_1(a,b;c;px)\,{}_2F_1(a',b';c';qx) = \sum_{n=0}^{\infty} (px)^n F_n,$$

I associate with it

$$z = \frac{bx}{c}\,{}_2F_1(a,b+1;c+1;px)\,{}_2F_1(a',b';c';qx) = \sum_{n=0}^{\infty} (px)^n G_n,$$

where F_n, G_n are as defined in (86) except that the ${}_4F_3$ now have

argument q/p instead of unity. We can then obtain a recurrence-relation

$$A_1 F_n - A_2 F_{n-1} + A_3 F_{n-2} = B_1 G_n - B_2 G_{n-1} + B_3 G_{n-2}$$

with one degree of freedom, which we assign to give the most convenient pair of differential equations in y, z. These are probably

$$\begin{aligned}(1-px)[\delta(\delta+c'-1)-qx(\delta+a')(\delta+b')]y+(a-1)bpx(1-qx)y \\ = (1-px)[(\delta+a-1)(\delta+a+c'-2)- \\ -qx(\delta+a+a'-1)(\delta+a+b'-1)]z-(a-1)(a-c)(1-qx)z \qquad (100)\end{aligned}$$

and

$$\begin{aligned}(b-c)[\delta(\delta+c'-1)-qx(\delta+a')(\delta+b')]y+ \\ +(a-1)px[(\delta+b)(\delta+b+c'-1)-qx(\delta+a'+b)(\delta+b'+b)]y \\ = (a+b-c-1)[\delta(\delta+c'-1)-qx(\delta+a')(\delta+b')]z+ \\ +(a-1)(b-1)[(2\delta+c+c'-2)-qx(2\delta+a'+b'+c-1)]z. \qquad (101)\end{aligned}$$

These are of respective ranks two and one in z, and their z-eliminant, i.e. the differential equation satisfied by $y = {}_2F_1 \times {}_2F_1$, can be obtained in the form (99), but it does not seem profitable to print it at length.

I hope to give further consideration to these extended functions in a later communication.

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FOURIER SERIES AND PRIMITIVE CHARACTERS

By A. P. GUINAND (*R.C.A.F.*)

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1. Introduction

It is known that there is a close connexion between Fourier series and Poisson's summation formula, and it has been shown that the latter can be generalized by the introduction of primitive characters as coefficients.*

This suggests that there may be a corresponding extension of Fourier series with primitive characters as coefficients. Such an extension is given in this paper. Any conditions for the validity of an ordinary Fourier expansion could readily be extended to cover the series with primitive characters, so the discussion is restricted to the simple case of continuous functions of bounded variation.

2. Preliminary Results

LEMMA 1. *If $\chi(n)$ is a character modulo k $(k > 1)$, then*

$$\chi(m)\chi(n) = \chi(mn),$$
$$\chi(m+rk) = \chi(m),$$

where k, m, n, and r are integers. Further, $\chi(n)$ vanishes if n is not prime to k.

LEMMA 2. *If $\chi(n)$ is a primitive character modulo k $(k > 1)$, then*

$$\bar{\chi}(m)\tau(k,\chi) = \sum_{r=1}^{k} \chi(r)e^{2\pi irm/k}$$

where
$$\tau(k,\chi) = \sum_{n=1}^{k} \chi(n)e^{2\pi in/k}$$

and $\bar{\chi}(m)$ is the conjugate of $\chi(m)$.

LEMMA 3.† *If $\chi(n)$ is a real primitive character modulo k $(k > 1)$, then $\chi(n) = \bar{\chi}(n)$ and*

(i) $\tau(k,\chi) = k^{\frac{1}{2}}$ *if* $\chi(-1) = 1$,

(ii) $\tau(k,\chi) = ik^{\frac{1}{2}}$ *if* $\chi(-1) = -1$.

* A. P. Guinand, *Annals of Math.* 42 (1941), 591–603.

† E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, I (Leipzig, 1909), chapters XXII and XXX. Lemma 3 follows by putting $\chi = \bar{\chi}$, $s = \frac{1}{2}$ in 497 (5).

LEMMA 4.* *If $f(x)$ is continuous and of bounded variation in the interval $-\frac{1}{2} < x < \frac{1}{2}$, then*

$$f(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{2\pi i n x},$$

where
$$c_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) e^{-2\pi i n t}\, dt.$$

Further, $c_n \to 0$ as $n \to \infty$ or $n \to -\infty$.

3. The extended Fourier series

Suppose $F(x)$ is a continuous function of bounded variation, and apply Lemma 4 to $F(x)e^{-2\pi i r x/k}$.

For $-\frac{1}{2} < x < \frac{1}{2}$

$$F(x)e^{-2\pi i r x/k} = \lim_{P\to\infty} \sum_{p=-P}^{P} c_p e^{2\pi i p x},$$

where
$$c_p = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i p t - 2\pi i r t/k} F(t)\, dt.$$

Hence

$$\begin{aligned} F(x) &= \lim_{P\to\infty} \sum_{p=-P}^{P} c_p e^{(pk+r)2\pi i x/k} \\ &= \lim_{P\to\infty} \sum_{p=-P}^{P} e^{(pk+r)2\pi i x/k} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-(pk+r)2\pi i t/k} F(t)\, dt. \end{aligned}$$

Hence, by Lemmas 1 and 2,

$$\begin{aligned} \bar{\chi}(m)\tau(k,\chi)F(x) &= F(x)\sum_{r=1}^{k} \chi(r) e^{2\pi i r m/k} \\ &= \sum_{r=1}^{k} \chi(r) e^{2\pi i r m/k} \lim_{P\to\infty} \sum_{p=-P}^{P} e^{(pk+r)2\pi i x/k} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-(pk+r)2\pi i t/k} F(t)\, dt \\ &= \lim_{P\to\infty} \sum_{r=1}^{k} \sum_{p=-P}^{P} \chi(pk+r) e^{(pk+r)(2\pi i x + 2\pi i m)/k} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-(pk+r)2\pi i t/k} F(t)\, dt \\ &= \lim_{P\to\infty} \sum_{n=-Pk+1}^{Pk+k} \chi(n) e^{2\pi i n(x+m)/k} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i n t/k} F(t)\, dt. \end{aligned}$$

Now, by Lemma 4, $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i n t/k} F(t)\, dt \to 0$

as $n \to \infty$ or $n \to -\infty$. Hence the above limit is equal to

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \chi(n) e^{2\pi i n(x+m)/k} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i n t/k} F(t)\, dt.$$

* Cf. E. C. Titchmarsh, *Theory of Functions* (Oxford, 1939), chapter XIII, or any text dealing with Fourier series.

Thus we have

THEOREM 1. *If $\chi(n)$ is a primitive character modulo k, and $F(x)$ is continuous and of bounded variation in $-\frac{1}{2} < x < \frac{1}{2}$, then in this range*

$$\bar{\chi}(m)F(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n\, \chi(n) e^{2\pi i n(x+m)/k},$$

where

$$c_n = \frac{1}{\tau(k,\chi)} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i n t/k} F(t)\, dt.$$

If $\chi(n)$ is a real primitive character, then we can use Lemma 3 to deduce the following results from Theorem 1.

THEOREM 2. *If $\chi(n)$ is a real primitive character modulo k ($k > 1$), $F(x)$ is continuous and of bounded variation in $-\frac{1}{2} < x < \frac{1}{2}$, and*

$$a_n = 2k^{-\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} F(t)\cos(2\pi nt/k)\, dt, \qquad b_n = 2k^{-\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} F(t)\sin(2\pi nt/k)\, dt,$$

then (i), *if* $\chi(-1) = 1$,

$$\chi(m)F(x) = \sum_{n=1}^{\infty} \chi(n)\{a_n \cos 2\pi n(x+m)/k + b_n \sin 2\pi n(x+m)/k\};$$

(ii), *if* $\chi(-1) = -1$,

$$\chi(m)F(x) = \sum_{n=1}^{\infty} \chi(n)\{a_n \sin 2\pi n(x+m)/k - b_n \cos 2\pi n(x+m)/k\}.$$

4. Examples

(i) If $\chi(n)$ is the real primitive character modulo 4, $\chi(1) = 1$, $\chi(3) = -1$, $\chi(2) = \chi(4) = 0$, and if $F(x)$ is continuous and of bounded variation in $-\frac{1}{2} < x < \frac{1}{2}$, then in this range

$$\chi(m)F(x) = \sum_{n=1}^{\infty} \chi(n)\{a_n \sin \tfrac{1}{2}n\pi(x+m) - b_n \cos \tfrac{1}{2}n\pi(x+m)\},$$

where

$$a_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(t)\cos \tfrac{1}{2}\pi nt\, dt, \qquad b_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(t)\sin \tfrac{1}{2}\pi nt\, dt.$$

(ii) If $\chi(n)$ is the real primitive character modulo 8, $\chi(1) = \chi(7) = 1$, $\chi(3) = \chi(5) = -1$, $\chi(2) = \chi(4) = \chi(6) = \chi(8) = 0$, and $F(x)$ satisfies the conditions of (i), then

$$\chi(m)F(x) = \sum_{n=1}^{\infty} \chi(n)\{a_n \cos \tfrac{1}{4}n\pi(x+m) + b_n \sin \tfrac{1}{4}n\pi(x+m)\},$$

where

$$a_n = \frac{1}{\sqrt{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} F(t)\cos \tfrac{1}{4}\pi nt\, dt, \qquad b_n = \frac{1}{\sqrt{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} F(t)\sin \tfrac{1}{4}\pi nt\, dt.$$

A NOTE ON FAREY SERIES

By P. ERDŐS (*Philadelphia*)

[Received 30 March 1943.]

[This note was received in the form of a letter addressed, through the *Quarterly Journal*, to the late Dr. Mayer. It has been put into its present form by the kindness of Professor Davenport.]

In extension of Dr. Mayer's theorems on the ordering of Farey series,* the following theorem can be proved:

THEOREM: *There exists an absolute constant c such that, if $n > ck$, and if*

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots$$

are the Farey fractions of order n, then $\dfrac{a_x}{b_x}$ *and* $\dfrac{a_{x+k}}{b_{x+k}}$ *are similarly ordered.*

Proof. As in Dr. Mayer's paper, we observe first that, if a_x/b_x and a_y/b_y (the latter being the greater) are not similarly ordered, then $a_y \geqslant a_x+1$, $b_y \leqslant b_x-1$, and therefore it suffices to prove that there are at least k Farey fractions between

$$\frac{a_x}{b_x} \quad \text{and} \quad \frac{a_x+1}{b_x-1}.$$

Case I. Suppose that $a_x/b_x < \frac{1}{6}$. In this case, we note that

$$\frac{a_x+1}{b_x-1} - \frac{a_x}{b_x} = \frac{a_x+b_x}{(b_x-1)b_x} > \frac{1}{b_x} \geqslant \frac{1}{n},$$

and we shall prove that there are at least k Farey fractions in the interval $\left(\dfrac{a_x}{b_x}, \dfrac{a_x}{b_x}+\dfrac{1}{n}\right)$. Let

$$\frac{a_x}{b_x}, \quad \frac{a_{x+1}}{b_{x+1}}, \quad \dots, \quad \frac{a_y}{b_y}$$

be the Farey fractions in this interval. Since the difference between two consecutive fractions is less than $\dfrac{1}{n}$, we have

$$\frac{1}{n} < \frac{a_{y+1}}{b_{y+1}} - \frac{a_x}{b_x} < \frac{2}{n}.$$

* A. E. Mayer, *Quart. J. of Math.* (Oxford), 13 (1942), 186–7, Theorems 1, 2.

If $n > 60$, it follows that $a_{y+1}/b_{y+1} < \frac{1}{6}+\frac{1}{30} = \frac{1}{5}$, so that $b_j \geqslant 6$ for $x \leqslant j \leqslant y+1$.

Now

$$\frac{a_{y+1}}{b_{y+1}}-\frac{a_x}{b_x} = \sum_{j=x}^{y}\left(\frac{a_{j+1}}{b_{j+1}}-\frac{a_j}{b_j}\right) = \sum_{j=x}^{y}\frac{1}{b_j b_{j+1}} < \sum_{j=x}^{y}\frac{2}{n\min(b_j, b_{j+1})},$$

since $b_j+b_{j+1} > n$. Thus

$$\Sigma \equiv \sum_{j=x}^{y}\frac{1}{\min(b_j, b_{j+1})} > \frac{1}{2}. \tag{1}$$

We write

$$\Sigma = \Sigma_1 + \Sigma_2, \tag{2}$$

where $\sum_1$ is extended over those values of j for which

$$\min(b_j, b_{j+1}) < 8k,$$

and $\sum_2$ over the others. Plainly

$$\Sigma_2 < \frac{y-x+1}{8k}.$$

If there is only one value of j (with $x \leqslant j \leqslant y+1$) for which $b_j < 8k$, then there are at most two terms in $\sum_1$, and, since $b_j \geqslant 6$, we have $\sum_1 \leqslant \frac{1}{3}$. If there are several such values of j, let them be $r_1, r_2, \ldots, r_t$. We have

$$\frac{2}{n} > \frac{a_{r_t}}{b_{r_t}}-\frac{a_{r_1}}{b_{r_1}} = \sum_{l=1}^{t-1}\left(\frac{a_{r_{l+1}}}{b_{r_{l+1}}}-\frac{a_{r_l}}{b_{r_l}}\right) \geqslant \sum_{l=1}^{t-1}\frac{1}{b_{r_l}b_{r_{l+1}}} > \frac{1}{8k}\sum_{l=1}^{t-1}\frac{1}{b_{r_l}}.$$

Hence

$$\sum_{l=1}^{t-1}\frac{1}{b_{r_l}} < \frac{16k}{n},$$

and the same holds for the sum from 2 to t. Thus

$$\sum_{l=1}^{t}\frac{1}{b_{r_l}} < \frac{32k}{n},$$

and, since each b_{r_l} occurs in at most two terms in $\sum_1$, it follows that

$$\Sigma_1 < \frac{64k}{n} < \frac{1}{3},$$

provided that $n > 192k$.

From (1) and (2), we have $\sum_2 > \frac{1}{6}$, that is

$$\frac{y-x+1}{8k} > \frac{1}{6}, \qquad y-x+1 > \tfrac{4}{3}k > k+1$$

for $k \geqslant 3$. This proves the result in Case I.

Case II. Suppose now that $a_x/b_x \geqslant \frac{1}{6}$. In this case,

$$\frac{a_x+1}{b_x-1}-\frac{a_x}{b_x}=\frac{a_x+b_x}{(b_x-1)b_x}>\frac{7}{6n}.$$

We shall prove that the interval

$$\left(\frac{a_x}{b_x},\frac{a_x}{b_x}+\frac{7}{6n}\right)$$

contains at least k Farey fractions. For this interval we have, in place of (1),

$$\sum_{j=x}^{y}\frac{1}{\min(b_j,b_{j+1})}>\frac{7}{12}.$$

If $b_j \geqslant 6$ for $x \leqslant j \leqslant y+1$, the proof of case (I) remains valid. Hence we can suppose that one of the b_j does not exceed 5. But, if $b_r \leqslant 5$, then

$$\frac{2}{n}>\left|\frac{a_j}{b_j}-\frac{a_r}{b_r}\right|\geqslant\frac{1}{5b_j}$$

for $j \neq r$, whence $b_j > \frac{1}{10}n > 40k$, provided that $n > 400k$. So every b_j except b_r satisfies $b_j > 40k$.

Since the difference between two consecutive Farey fractions is at most $1/2(n-1)$, we have (omitting in the summations $j=r$ and $j+1=r$)

$$\sum_{j=x}^{y}{}'\left(\frac{a_{j+1}}{b_{j+1}}-\frac{a_j}{b_j}\right)>\frac{7}{6n}-\frac{2}{2(n-1)}>\frac{1}{10n}.$$

Hence
$$\frac{1}{10n}<\sum_{j=x}^{y}{}'\frac{1}{b_j\,b_{j+1}}<\frac{2}{n}\sum_{j=x}^{y}{}'\frac{1}{\min(b_j,b_{j+1})},$$

whence
$$\sum_{j=x}^{y}{}'\frac{1}{\min(b_j,b_{j+1})}>\frac{1}{20}.$$

Since $\min(b_j,b_{j+1})>40k$ in this sum, we have

$$\frac{y-x+1}{40k}>\frac{1}{20},\qquad y-x+1>2k\geqslant k+1.$$

This completes the proof.

I have not been able to find the best possible value for the constant c in the above result. It is easy to prove the following results, which are closely connected with that proved above:

(i) To every $\epsilon>0$ there exists a $c=c(\epsilon)$ such that any interval of length $(1+\epsilon)/n$ contains at least cn Farey fractions of order n.

(ii) If $f(n) \to \infty$ as $n \to \infty$, any interval of length $n^{-1}f(n)$ contains

$$\frac{3}{\pi^2} nf(n) + o(nf(n))$$

Farey fractions of order n.

It may be of interest to remark that Lemma 1 of Dr. Mayer's paper can be strengthened as follows: There exists a constant c_1 such that any interval of length $L = k^{c_1}$ contains a set of at least k mutually prime integers. This can be proved by Brun's method. It would be interesting to have a good estimate for the best possible value $L(k)$ of L from below. It follows from a result of Rankin* that

$$L(k) > c_2 \frac{k \log k \log \log \log k}{(\log \log k)^2}.$$

* *J. of London Math. Soc.* 13 (1938), 242.

THE MAXIMUM NUMBER OF LINES LYING ON A QUARTIC SURFACE

By B. SEGRE (*Manchester*)

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1. THE general algebraic surface of order n contains no lines, if $n \geqslant 4$. But obviously there are special non-singular, and therefore non-ruled, surfaces of any given order $n \geqslant 4$, which contain some lines. The question of determining the maximum number of lines possessed by such surfaces has been often indicated,* but never solved. In this paper I shall establish that:

The maximum number of lines lying on a non-singular quartic surface is 64.

Since a quartic surface with 64 lines has long ago been discovered by F. Schur,† I need only to show that no more than 64 lines can lie on any non-singular quartic surface. I succeed in proving this in § 9, by the use of certain general properties on surfaces containing lines (§§ 2–8).‡ Some of these properties concern quartic surfaces only. Others are established for surfaces of arbitrary order, and lead to an upper limit (if not to the maximum) for the number of lines lying on a non-singular surface of order $n > 4$ (§ 4).

2. Let F_n be a non-singular surface of order $n \geqslant 3$, containing a line C. A plane π passing through C intersects F_n residually in a curve Γ, of order $n-1$. When π varies in the pencil of axis C, the curve Γ describes on F_n a pencil $|\Gamma|$ free from base-points. For a base-point of $|\Gamma|$ would lie on C, and be a multiple point of F_n. Hence no curve Γ contains C as a component, and $|\Gamma|$ cuts out on C a g^1_{n-1} free from fixed points. I shall prove that:

Either each curve Γ *intersects* C *in* $n-1$ *points which are inflexions for* Γ, *or the points of* C *each of which is an inflexion for a curve* Γ *are* $8n-14$ *in number.*

* Cf. e.g. W. Fr. Meyer, 'Flächen vierter und höherer Ordnung': *Encykl. Math. Wiss.*, Bd. 3, Teil 2, 2. Hälfte B, 1533–1779, § 54.

† F. Schur, *Math. Ann.* 20 (1882), 84.

‡ I shall utilize part of these properties in another paper, dealing with arithmetical properties of quartic surfaces, where I shall incidentally obtain a new non-singular quartic surface containing 64 lines.

Let us take C as the fundamental line $x_3 = x_4 = 0$, so that the equation of F_n can be written in the form

$$x_3\alpha_0(x_1,x_2)+x_4\alpha_1(x_1,x_2)+x_3^2\beta_0(x_1,x_2)+x_3x_4\beta_1(x_1,x_2)+x_4^2\beta_2(x_1,x_2)+$$
$$+x_3^3\gamma_0(x_1,x_2,x_3,x_4)+x_3^2x_4\gamma_1(x_1,x_2,x_3,x_4)+x_3x_4^2\gamma_2(x_1,x_2,x_3,x_4)+$$
$$+x_4^3\gamma_3(x_1,x_2,x_3,x_4) = 0,$$

where the α's, β's, γ's are forms of degrees $n-1$, $n-2$, $n-3$ respectively in their arguments. The curve Γ intersected on F_n by the plane π of equation $x_4 = \lambda x_3$ is represented by this equation and

$$\alpha_0+\lambda\alpha_1+x_3(\beta_0+\lambda\beta_1+\lambda^2\beta_2)+x_3^2(\bar{\gamma}_0+\lambda\bar{\gamma}_1+\lambda^2\bar{\gamma}_2+\lambda^3\bar{\gamma}_3) = 0, \tag{1}$$

where $\bar{\gamma}_0$, $\bar{\gamma}_1$, $\bar{\gamma}_2$, $\bar{\gamma}_3$ are the results of substituting λx_3 for x_4 in γ_0, γ_1, γ_2, γ_3 respectively. The set of g^1_{n-1} cut out by Γ on C is given by the equations $x_3 = x_4 = 0$ and

$$\alpha_0(x_1,x_2)+\lambda\alpha_1(x_1,x_2) = 0. \tag{2}$$

The condition for a point of this set to be an inflexion (or, in particular, a multiple point) of Γ, is that the Hessian of (1) must vanish there. After writing 0 for x_3, the condition takes the form

$$\begin{vmatrix} a_1 & b_1 & d_2 \\ b_1 & c_1 & e_2 \\ d_2 & e_2 & f_3 \end{vmatrix} = 0, \tag{3}$$

where a_1, b_1, c_1, d_2, e_2, f_3 are polynomials in λ of degrees equal to the respective indices, the coefficients of which are forms of degree $n-3$ in x_1, x_2. Now, either (2) implies (3) for every λ, or the elimination of λ from (2), (3) gives a homogeneous equation of degree

$$5(n-1)+3(n-3) = 8n-14$$

in x_1, x_2. In the first case, each curve Γ intersects C in $n-1$ points which are inflexions of Γ. In the second case, the equation obtained above gives the points of C each of which is an inflexion for a curve Γ, so that the number of such points is in fact $8n-14$.

The first case cannot arise for $n = 3$, since otherwise each curve Γ would break up into two lines, and F_3 would be a ruled surface. Hence the theorem then reduces to the well-known fact that, on a non-singular F_3, each line is met by 10 other lines lying on F_3. For $n \geqslant 4$ we call C a *line of the first* or *of the second kind* of F_n, according as C presents the second or the first of the two cases distinguished before. We shall determine later on all the quartic surface containing a line of the second kind (§ 6).

As an immediate corollary of the previous theorem we see that:

A line of the first kind of a non-singular surface F_n, of order $n \geqslant 4$, is met by no more than $8n-14$ lines lying on F_n.

3. A non-singular F_n of order $n \geqslant 3$ is non-ruled, and so it possesses a *flecnodal curve*, K say, the locus of the points of contact of F_n with the tangents having a four-point contact. This locus is of order $11n^2-24n$, and can be obtained as complete intersection of F_n with a surface of order $11n-24$.* It is obvious that, if F_n contains a line C, then C is a component of K. If c ($\geqslant 1$) is the multiplicity of C for K, the intersection number of C with the residual curve $K-cC$ is

$$[C, K-cC] = [CK]-c[C^2] = 11n-24+c(n-2), \tag{4}$$

since, as is easily seen, C has the virtual degree $[C^2] = -(n-2)$.

We now proceed to investigate when a point P of C is a multiple point of K. Let us choose non-homogeneous coordinates (x, y, z), such that P is the origin, C is the x-axis, and the tangent plane π of F_n at P has the equation $z = 0$. Then, in the neighbourhood of P, F_n is represented by an equation of the form

$$z = f(x,y), \tag{5}$$

where $$f(x,y) = y(hx+ky+px^2+qxy+ry^2+sx^3+...) \tag{6}$$

is a power series convergent in the neighbourhood of $x = y = 0$. Moreover, the equations of K are (5) and

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & 2\frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & 0 & 0 \\ 0 & \frac{\partial^2 f}{\partial x^2} & 2\frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & 0 \\ 0 & 0 & \frac{\partial^2 f}{\partial x^2} & 2\frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \\ \frac{\partial^3 f}{\partial x^3} & 3\frac{\partial^3 f}{\partial x^2 \partial y} & 3\frac{\partial^3 f}{\partial x \partial y^2} & \frac{\partial^3 f}{\partial y^3} & 0 \\ 0 & \frac{\partial^3 f}{\partial x^3} & 3\frac{\partial^3 f}{\partial x^2 \partial y} & 3\frac{\partial^3 f}{\partial x \partial y^2} & \frac{\partial^3 f}{\partial y^3} \end{vmatrix} = 0.\dagger \tag{7}$$

* Cf. e.g. G. Salmon, *Analytic Geometry of Three Dimensions*, vol. ii (5th ed., 1915), 278.

† For, if $Q(x,y,z)$ is a point of F_n near P, we can represent F_n in the neighbourhood of Q as locus of the points (X, Y, Z) satisfying $Z = f(X,Y)$. On

The first column of the determinant is divisible by y, as we see on substituting for the derivatives of $f(x, y)$ from (6); this is in accordance with the fact that K contains C at least once. The constant term of the expansion which we deduce from (7) after having suppressed the factor $2^3.6^2.y$, i.e.

$$\begin{vmatrix} p & h & k & 0 & 0 \\ 0 & 0 & h & k & 0 \\ 0 & 0 & 0 & h & k \\ s & p & q & r & 0 \\ 0 & 0 & p & q & r \end{vmatrix} = \begin{vmatrix} p & h \\ s & p \end{vmatrix} . \begin{vmatrix} h & k & 0 \\ 0 & h & k \\ p & q & r \end{vmatrix},$$

obviously vanishes if and only if P is a multiple point of K. Hence this occurs if either

$$\begin{vmatrix} h & k & 0 \\ 0 & h & k \\ p & q & r \end{vmatrix} = 0, \tag{8}$$

or

$$p^2 - hs = 0. \tag{9}$$

Now the curve Γ, intersected on F_n by π apart from C, has in the neighbourhood of P the equations

$$z = 0, \qquad (hx+ky)+(px^2+qxy+ry^2)+sx^3... = 0.$$

Hence (8) expresses the necessary and sufficient condition for P to be *an inflexion* (or, in particular, a multiple point) of Γ.

Next we suppose that P is a simple point, but not an inflexion, of Γ. Moreover, we assume that Γ does not touch C at P, i.e. that $h \neq 0$, since otherwise (9) could not hold; for $h = 0$ and (9) would imply $p = 0$, and so P would be an inflexion of Γ. Then the quadrics

expanding $f(X, Y)$ in the neighbourhood of (x, y) with Taylor's formula, we can write this equation in the form

$$\zeta = \phi_1(\xi, \eta)+\phi_2(\xi, \eta)+\phi_3(\xi, \eta)+...,$$

where $\zeta = Z-z$, and ϕ_i is a form of degree i in $\xi = X-x$ and $\eta = Y-y$ ($i = 1, 2,...$). With this notation, and on interpreting X, Y, Z as coordinates of a point-variable in space (and not restricted to lie on F_n), we see that $\zeta = \phi_1(\xi, \eta)$ is the plane χ touching F_n at Q, while on χ the curve of intersection of F_n and χ is represented by $\phi_2(\xi, \eta)+\phi_3(\xi, \eta)+... = 0$. Hence the principal tangents of F_n at Q are given on χ by $\phi_2(\xi, \eta) = 0$, and one of these two lines has a four-point contact with F_n at Q (i.e. Q lies on K), if and only if it also satisfies the equation $\phi_3(\xi, \eta) = 0$. The curve K is consequently represented on F_n by (7), since this is the result of the elimination of ξ and η from the last two equations.

passing through C and intersecting F_n residually in curves Δ having a cusp at P, with C as the cuspidal tangent, constitute the pencil

$$h^3xy+h^2ky^2+hpxz+(hq-kp)yz-h^2z+\lambda z^2 = 0,$$

where λ is a parameter. The condition (9) holds if and only if the curves Δ have *a tacnode* at P.

On recalling § 2 and (4), we see that:

A line C of F_n which is a simple component of the flecnodal curve K is always of the first kind. C then meets the residual curve $K-C$ in $12n-26$ *points, of which* $8n-14$ *are of the first of the two types considered above, while the remaining* $4(n-3)$ *points are of the second type.*

4. There are on C no points of the second type, if $n = 3$. This agrees with the obvious fact that the flecnodal curve (of order 27) of a non-singular cubic surface F_3 consists of the 27 lines lying on F_3, each counted once. From § 3, however, we deduce that a similar fact does not hold for any F_n of order $n \geqslant 4$, and that:

The number of lines lying on a non-singular F_n of order $n \geqslant 4$ cannot exceed $(n-2)(11n-6)$.

For, if each line of F_n is a multiple component of the flecnodal curve K, the number of these lines does not exceed

$$\tfrac{1}{2}(11n^2-24n) < (n-2)(11n-6).$$

If, on the other hand, K contains at least one line C of F_n as a simple component, then K, in addition to the lines of F_n, must contain a curve meeting C in $4n-12$ points. The order of this curve is consequently not less than $4n-12$, and so the number of lines of F_n cannot exceed

$$11n^2-24n-(4n-12) = (n-2)(11n-6).$$

This result could probably be improved by means of some extension of the argument we shall use in § 9 to establish (as stated in § 1) the best possible result for $n = 4$.

An improvement can be easily obtained for arbitrary n, if we restrict ourselves to surfaces F_n containing no triplets of concurrent lines and possessing a flecnodal curve free from multiple components. Then F_n contains only a finite number of points each of which is a biflecnode for the section of F_n with the respective tangent plane; and the number β of such points is

$$\beta = 5n(7n^2-28n+30).*$$

* Cf. G. Salmon, op. cit. 292.

Let us denote by ν the number of lines lying on F_n, and by α the number of pairs of incident lines of F_n. Then

$$\alpha \leqslant \beta,$$

since a point of intersection of two lines of F_n is one the β points considered above, and

$$\beta \geqslant (8n-14)\nu-\alpha,$$

since each line of F_n contains $8n-14$ of these β points (§ 2). Hence

$$(8n-14)\nu \leqslant \alpha+\beta \leqslant 2\beta = 10n(7n^2-28n+30),$$

and so

$$\nu \leqslant 5n(7n^2-28n+30)/(4n-7).$$

This inequality gives $\nu \leqslant 66$ for $n = 4$, $\nu \leqslant 125$ for $n = 5$; and, for $n \geqslant 4$, is *stronger* than the previous inequality $\nu \leqslant (n-2)(11n-6)$. For $n = 3$ we have $\nu = 27$, and both inequalities reduce to $\nu \leqslant 27$.

5. With the notation of § 2 we shall establish the following theorem:

If C is a line of the second kind of F_n, then a curve Γ of the corresponding pencil $|\Gamma|$ having at least two coinciding intersections with C at a point P, has either a double point at P, or four intersections with C at P.

Let us suppose that P is a simple point of Γ. Then Γ has an inflexion at P, since C is a line of the second kind of F_n, and we can represent F_n as in § 3 with (5), (6), where now

$$h = 0, \qquad k \neq 0, \qquad p = 0.$$

The theorem will follow, if we prove that $s = 0$.

For this purpose we consider on C a point $P_1(\sigma, 0, 0)$ near P, and express the condition (implied by the hypothesis that C is a line of the second kind of F_n) that the curve Γ_1 of $|\Gamma|$ containing P_1 has an inflexion at this point. The equation of the plane touching F_n at P_1 is

$$z = (s\sigma^3+\ldots)y,$$

where the coefficient of y is a power series in σ, convergent in the neighbourhood of $\sigma = 0$. Hence Γ_1 is represented, in the neighbourhood of P_1, by this equation and

$$ky+qxy+ry^2+sx^3+\ldots = s\sigma^3+\ldots.$$

On putting $x = x_1+\sigma$, we obtain

$$h_1x_1+k_1y+p_1x_1^2+q_1x_1y+r_1y^2+s_1x_1^3+\ldots = 0,$$

where

$$h_1 = 3s\sigma^2+\ldots, \qquad k_1 = k+q\sigma+\ldots;$$

$$p_1 = 3s\sigma+\ldots, \qquad q_1 = q+\ldots, \qquad r_1 = r+\ldots,$$

and so the above condition gives

$$\begin{vmatrix} h_1 & k_1 & 0 \\ 0 & h_1 & k_1 \\ p_1 & q_1 & r_1 \end{vmatrix} = 3sk^2\sigma + \ldots = 0.$$

Hence $s = 0$, since this equation must be an identity in σ, and $k \neq 0$.

6. From now on, we shall restrict ourselves to the case $n = 4$. If C is a line lying on a non-singular quartic surface F, the pencil $|\Gamma|$ of cubics cut out on F by the planes through C contains four curves having two coinciding intersections with C. When a point P describes C, the tangent G at P to the cubic Γ containing this point generates a ruled surface R, and comes four times into coincidence with C. Hence C is for R a directrix line of multiplicity $1+4 = 5$. Since a general plane passing through C meets R, residually to C, in three generators, we see that:

The ruled surface R is of order $5+3 = 8$, and contains C with multiplicity 5.

If a curve Γ acquires a double point at a point P of C, then the plane of Γ becomes a component of R, since all the lines of this plane passing through P are in this case lines G.

Let us now in particular suppose that *C is a line of the second kind of F*. Then, from § 5, each of the four cubics Γ having two coinciding intersections with C, either has a double point lying on C, or intersects C at four coinciding points. From § 2 we see that the second case must now be ruled out, since it would imply that C is a component of Γ. Hence each of the above cubics Γ has a double point on C, so that R contains the four planes of these cubics, and consists residually of *a ruled surface Φ, which is of order* 4 *and has C as a simple directrix line.*

The equation of such a surface Φ can be easily obtained, on remarking that Φ is the dual of a quartic surface with a triple line.

Another simple way of determining these surfaces is as follows. The ruled surface Φ is unicursal, and so irreducible, since we obtain a $(1, 1)$-correspondence between the generators G of Φ and the points P of C, if we associate G and P when P lies on G. The section of Φ with a general plane is consequently a unicursal plane quartic curve, which therefore has either a triple point, or three non-collinear double points.

In the first case, the surface Φ has a triple line, T say, skew to C, since otherwise Φ would be reducible. Then Φ appears as *the locus of the lines joining corresponding points of two skew lines C, T, related by a* $(3, 1)$ *algebraic correspondence.* Conversely, such a locus is a quartic ruled surface, having C as a simple directrix line.

In the second case, the surface Φ has a twisted double cubic curve, D say, as locus of multiple points. From the irreducibility of Φ it follows that D is irreducible, and has no points in common with C. Moreover, from a well-known general property of ruled surfaces, each generator of Φ must contain two points of D. Hence Φ is *the locus of the chords of an irreducible twisted cubic D which are incident with a line C having no points in common with D.* Conversely, such a locus is a quartic ruled surface, having C as a simple directrix line.

Let us now return to the non-singular quartic surface F, possessing a line C of the second kind, and to the ruled quartic surface Φ defined as before by F and C. The general generator G of Φ meets C at a point P, which absorbs all the four intersections of G and F. Hence the intersection of F and Φ contains C with multiplicity four. Moreover, if Q is an arbitrary point common to F, Φ and not lying on C, then any generator G of Φ passing through Q must be completely on F, since G intersects F at Q and at $P = CG$ counted four times. It follows that the intersection of F, Φ residual to C consists of $4.4-4 = 12$ lines, generators of Φ, distributed three by three in four planes passing through C.

The quartic surface Ψ formed by these planes and the quartic surface Φ cut out on F the same curve of the 16th order, consisting of the above 12 lines and of C counted four times. Hence F belongs to the pencil of Φ and Ψ. Let us conversely consider the surface F represented by

$$\Phi+\lambda\Psi = 0, \tag{10}$$

where λ is a parameter, $\Phi = 0$ is the equation of a ruled quartic surface Φ possessing a simple directrix line C, and $\Psi = 0$ is the equation of a quartic surface Ψ consisting of four planes passing through C. These planes must be distinct, and none of them can touch Φ along a generator, if we wish F to be non-singular. Conversely, if the four planes Ψ are distinct, and none of them touches Φ along a generator, then, from Bertini's theorem, we see that F is non-singular by a general choice of λ. Then each generator G of Φ, not lying on Ψ, has its four intersections with F coinciding with the

point of intersection of G and C. Hence C is a line of the second kind on F, and so (10) *represents the most general non-singular quartic surface containing a line of the second kind.*

7. We can now prove the following theorem:

If a non-singular quartic surface F contains a line C, then F may contain at most 18 *lines incident with C.*

This is included in the final result of § 2, if C is a line of the first kind. We can therefore confine ourselves to the case in which C is a line of the second kind of F, so that F can be represented by the equation (10). From § 6 we know that F contains 12 distinct lines incident with C, constituting the intersection of Φ and Ψ residual to C. The question is to decide how many additional lines incident with C may lie on F.

If F contains such a line, L say, then L cannot lie on either Φ or Ψ, since otherwise, from (10), L would lie on both these surfaces. Moreover, also from (10), we see that these two surfaces cut out on L the same set of four points. As Ψ consists of four planes through C, this set reduces to the intersection point of L and C, say O, counted four times. Since the plane π of L and C intersects Φ along C (counted once) and three generators, none of which may coincide with C (§ 6), it follows that each of these three generators contains O. Hence these three lines must coincide with the unique generator of Φ passing through O. Conversely, if a plane π through C intersects Φ again in a generator G to be counted three times, then π is not a component of Ψ (§ 6). The intersections of Φ and Ψ with π being $C+3G$ and $4C$ respectively, we see from (10) that π intersects on F the line C and three further distinct lines belonging to the pencil of C and G. Hence the required additional lines of F distribute in sets of three lines, each set lying in a plane through C which osculates Φ along a generator.

We shall show that the number of such planes cannot exceed two, and then the theorem will follow at once. If a plane through C osculates Φ along a generator G, the point of intersection of C and G is a triple point of the g_3^1 cut out on C by the pencil $|\Gamma|$ (§§ 2, 6). Hence there are in fact no more than two such planes, since a g_3^1 on a line can have no more than two triple points. From § 6 and the previous discussion we see that:

If a non-singular quartic surface F contains a line C of the second kind, then F contains either 12, *or* 15, *or* 18 *lines incident with C, according as the g_3^1 determined by F on C has* 0, *or* 1, *or* 2 *triple points.*

8. We complete the results of § 7 by showing that:

If a non-singular quartic surface F contains a line C, and more than 12 *lines incident with C lie on F, then three among these lines are in a plane passing through C.*

The pencil $|\Gamma|$ cut out on F by the planes through C has no base points (§ 2), and is clearly of genus $p = 1$; moreover, none of its curves may have a multiple component, since F is non-singular. Hence the curves of $|\Gamma|$ with a double point are $I+4p = 24$ in number, where $I = 20$ is the Zeuthen-Segre invariant of F.* A line of F incident with C is a component of a curve Γ, which consists either of this line and an irreducible conic, or of three distinct lines. In the first case Γ absorbs two of the 24 singular curves of $|\Gamma|$, so that this case cannot occur more than 12 times. Hence the second case must in fact occur at least once, if F contains more than 12 lines incident with C.

9. We are finally in a position to prove that:

The number of lines lying on a non-singular quartic surface F cannot exceed 64.

If four of the lines of F, say A, B, C, D, are in a plane π, then A, B, C, D constitute the complete intersection of F and π. Hence each of the remaining lines of F meets π at a point of this intersection, and so is incident with one of the lines A, B, C, D. Each of these four lines is incident with the remaining three, and with at most 15 further lines of F (§ 7). It follows that the total number of lines lying on F is then at most

$$4+4.15 = 64.$$

Let us now suppose that no four lines of F are coplanar. This means that the plane π joining any two incident lines of F, say C, D, meets F residually in an irreducible conic, Ω say. If this is not a component of the flecnodal curve K of F, then from § 3 we see that K meets Ω in 40 points. Four of these points are the intersections of Ω with $C+D$. The remaining 36 include all the intersections of Ω with the lines of F meeting neither C nor D, since each of these

* Cf. C. Segre, *Atti Acc. Scienze Torino*, 31 (1896), § 4.

lines is a component of K, and meets π at a point lying on Ω. Such a point has at least multiplicity h in the above set of 36 points, if it lies on h ($\geqslant 1$) lines of F. Moreover, from § 8, each of the lines C, D is incident with the other, and with at most 11 further lines of F. Hence, in present circumstances, the number of lines lying on F cannot exceed

$$2+11+11+36 = 60.$$

We can therefore suppose that the plane of any two incident lines of F (if any) meets F again in an irreducible conic, which is a component of K. If the number of pairs of incident lines of F exceeds 7, then K contains at least 8 irreducible conics, in addition to the lines of F. Since K is of order 80, the number of these lines cannot exceed

$$80-2.8 = 64.$$

We can consequently confine ourselves to the case in which F contains no more than 7 pairs of incident lines. There is then at least one line of F, say C, incident with no more than 6 of the remaining lines of F (possibly with none of them). We denote by H the curve consisting of these lines (if any), each taken once, and of C and the lines of F skew to C, each taken with the multiplicity it has as a component of K. Hence $K-H$ is an effective curve not containing C as a component, and $[H, C] \leqslant 6-2 = 4$, since C has the virtual degree -2. From $[K, C] = 20$ (§ 3), it follows $[K-H, C] \geqslant 16$, so that the order of $K-H$ cannot be less than 16. Therefore the order of H, and *a fortiori* the number of lines of F, cannot exceed

$$80-16 = 64.$$

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